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Consistent Wave Equations for Families of Massive Particles with Any Spin

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We investigate all finite-component invariant wave equations $(-i\beta^{\mu}\partial_{\mu}+\kappa)\psi=0$ that have a complete set of solutions describing massive particles. We discuss a practical method for the computation of the mass spectrum, an appropriate scalar product and adjoint wave function (defined without a hermitizing matrix), and the discrete symmetries P, T, and C. The Klein–Gordon divisor is studied in some detail; the corresponding propagators are found to be without contact terms. Such wave equations seem to offer the basis for a consistent description of multiplets of particles with any spin: They lead to quantum field theories that can be derived from a Lagrangian, have positive definite metric and energy, and satisfy the canonical commutation relations. Although we are here only considering noninteracting theories, it is evident that such equations are free of the inconsistencies usually encountered in higher spin theories.

1. INTRODUCTION

Fifty years after the discovery of the Dirac equation (Dirac, 1928a, b), and with many hundreds of articles and numerous books (Corson, 1953; Corben, 1968; Takahashi, 1969; Visconti, 1969; Paërl, 1969; Velo and Wightman, 1978) about relativistic wave equations, yet another series of papers on this same subject calls for a special explanation. Our reason for the renewed investigation of this topic is quite simple: Despite the enormous literature in this field only partial answers are known to many questions, and most problems of practical interest have not been studied at all. For example, it has been known for a long time that there are wave equations other than the usual Klein–Gordon or Dirac equation which describe a whole spectrum of particles, and it also was suspected that the different types of wave equations might be dynamically inequivalent once an interaction was turned on. However, no serious attempts have been

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made to determine the physical content of more general wave equations, and, except for only a few special cases, no explicit calculations of mass spectra, magnetic moments, form factors, and cross sections have been carried out for such theories, not even for those with only a finite number of components. By a (free) relativistic wave equation we shall mean in the following a set of first-order differential equations of the type (in units $\hbar = c = 1$, and the metric $1, -1, -1, -1$)

$$(-i\beta^\mu \partial_\mu + \kappa)\psi(x) = 0 \tag{1.1}$$

Any free wave equation of higher order can always be brought into this form by introducing a suitable number of new components in the wave function; furthermore, this reduction can be accomplished without destroying the assumed manifest covariance of the theory. [Compare in this respect the Feshbach–Villars (Feshbach and Villars, 1958) and the Duffin–Kemmer–Petiau (Petiau, 1936; Duffin, 1938; Kemmer, 1939) first-order versions of the Klein–Gordon equation.] The wave function $\psi(x)$ is assumed to transform locally under the Poincaré group,

$$\psi'(x') = D(g)\psi(x) \tag{1.2}$$

with $x' = L(g)x + a$, and $D(g)$ being a certain representation of the quantum mechanical restricted Lorentz group $SL(2,C)$. [For no apparent reason space–time translations are usually represented trivially, i.e., $D(a,g) = D(g)$.] The matrices β^μ and κ are required to satisfy

$$D^{-1}(g)\beta^\mu D(g) = L(g)^\mu{}_\nu \beta^\nu \tag{1.3a}$$

$$D^{-1}(g)\kappa D(g) = \kappa \tag{1.3b}$$

The Dirac equation is certainly the best known and most successful example of such a wave equation, giving an accurate description of the properties of leptons. It never has been seriously investigated whether any of the more general wave equations (1.1) could be used for the phenomenological description of the strongly interacting particles.

At first it is quite surprising that after such a long time almost nothing is known about the physical content of these wave equations. This appears to be largely due to the peculiarities of the historical development. In the early thirties and forties, when there was general interest in relativistic wave equations and field theory, only a few fundamental particles were known. These particles were considered to be the elementary building blocks of matter, and it was natural to assume that only the most simple types of wave functions should be used for their theoretical description. In addition to this philosophical prejudice based on incomplete experimental information, there were also technical difficulties: The multidimensional matrices β^{μ} and κ appearing in such wave equations make explicit calculations of even the most elementary physical observables quite difficult. As at that time there were no compelling reasons, based on experiment, for a thorough investigation of these general wave equations, only some of their more formal aspects have been studied (Corson, 1953; Gårding, 1944; Gel'fand, Minlos, and Shapiro, 1963; Naimark, 1964). By and large, such theories were considered a mere mathematical curiosity and, as regards actual calculations of physical quantities, they were treated with benign neglect. On the other hand, in the fifties and sixties, when there occurred the dramatic increase of experimental information about the spectrum of fundamental particles and their properties, wave equations and field theory had fallen into temporary disrepute, and it was widely held that only a pure S-matrix theory would be capable of describing the interactions of the various fundamental particles.

Today, as the solution of the universal, all-embracing bootstrap seems to be less imminent than it appeared a decade ago, field theoretical methods are coming into the forefront again. [With no claim to completeness, see the following references for a small sample of the numerous recent studies of more general wave equations and field theories: Joos (1962), Wichmann (1962), Weinberg (1964a, b; 1968), Pursey (1965), McKerrell (1966), Niederer and O'Raifeartaigh (1974a, b), Krajcik and Nieto (1977),[1] Singh and Hagen (1974a, b), Barut and Wilson (1976), Fischbach, Nieto, Primakoff, and Scott (1974a), Jenkins (1972a), Allcock (1975a, b), Allcock and Hall (1977, 1978), Shamaly and Capri (1972, 1973), and Hurley and Sudarshan (1974).] All our present experimental information indicates a rather complicated internal structure of the strongly interacting particles, in contrast to the leptons. Hence there is no longer any reason why only the most simple types of wave functions should be used for the field theoretical description of these particles. Whereas the Dirac equation gives an accurate account of the leptons but fails for the strongly interacting particles, it may very well be that certain more general wave equations can be used to describe the structure of the hadrons as observed in their electromagnetic and weak interactions. What is still missing for this purpose is a detailed and systematic investigation of the general wave equation (1.1) and its physical content.

Recently, we have computed some simple physical observables for a large class of finite-component wave equations. We have found a practical way to determine the mass spectra of such wave equations (Biritz, 1975a);

[1]This is the last of seven papers on the Bhabha equation; references to the earlier parts may be found in this article.

for a simple example see Biritz (1975b). For these wave equations we have also evaluated two important static observables sensitive to an internal structure, namely, the magnetic moments predicted for minimal coupling (Biritz, 1975a, c) and the renormalization of the axial-vector coupling constant in beta decay (Biritz, 1975d). We have found these equations to be dynamically inequivalent, and that the spin alone does not determine the physical properties of a particle described by such a wave equation. These properties do not only depend on the "type" of wave function employed, i.e., on its specific transformation law $D(g)$ under the Lorentz group, but also on the details of the full mass spectrum of the wave equation. It has been said (Weinberg, 1964a, p. 1319) that "a free-field equation is nothing but an invariant record of which components are superfluous." This is not true. In general, all these wave equations describe not just a single particle but a whole spectrum of states with different masses, spins, and parities. All the states of such an equation are of equal importance and they cannot all be ignored save one, as their presence affects, even in lowest-order perturbation theory, the physical properties of any particle in the spectrum. In view of the observed spectrum of hadrons and their excited states we consider this a blessing, not a curse; we do not try to project out a single state but we are deliberately looking for equations that describe more than one particle. What makes these equation so attractive is not so much the fact that they contain particles of higher spin but that they provide new, inequivalent descriptions for particles of spin zero and one-half. In particular, we obtained g factors significantly different from 2 even for minimal coupling (Biritz, 1975a, c), and also a nontrivial renormalization of the axial-vector coupling constant (Biritz, 1975d). Hence such wave equations seem to offer the possibility of describing in a phenomenological way particles with an internal structure like the proton, using as input the masses, spins, and parities of its excited states. Physically this seems to be quite plausible as we expect the interaction mechanism that gives rise to the nucleon resonances also to be responsible for the structure of the nucleon as observed in its electromagnetic and weak interactions.

Up to now the wave equations corresponding to various particles were selected mainly on the basis of their mathematical simplicity, like least number of components of the wave function consistent with a given spin. This is not a very reasonable criterion: We have learned that the physical properties of a particle are not solely determined by its spin but also depend drastically on the type of wave function chosen, and on the full mass spectrum of the wave equation. The problem of describing interactions in relativistic field theory consists therefore not only of coupling various fields together in a Lorentz invariant way, but first, and perhaps most important of all, of selecting the correct type of wave function for a given particle. This choice is ultimately determined by experiment and not by reasons of mathematical convenience; in addition to the spin of the particle we need further experimental information like the spectrum of its excited states.

Besides being of obvious practical interest there are also purely theoretical reasons for a second look at more general wave equations. Beginning already with the very first papers on relativistic wave equations for particles of higher spin, several difficulties have been encountered. For example, the equations proposed by Dirac (1936) were shown to be inconsistent in the presence of a minimal electromagnetic coupling by Fierz and Pauli (1939). More recently, Schwinger (1963) noticed difficulties with the canonical quantization of theories with spin $\geqslant 3/2$. Johnson and Sudarshan (1961) pointed out that the canonically quantized Rarita–Schwinger and Bhabha fields for spin $3/2$ no longer obey the proper anticommutation relations (i.e., zero at spacelike distances) when minimally coupled to an external electromagnetic field. Velo and Zwanziger (1969a, b) discovered that this difficulty already shows up in the corresponding c-number theory, independent of the quantization procedure employed. Among other examples they showed that the Rarita–Schwinger equation for spin $3/2$ becomes acausal for minimal coupling, no matter how weak the external field. There are also other types of instabilities. For a spin-$3/2$ particle in a constant magnetic field Seetharanam, Prabhakaran, and Mathews (1975) found normal modes which cease to be real once the strength of the field exceeds a certain critical value. Wightman (1976) has demonstrated for two classes of wave equations that even with minimal coupling the ingoing and outgoing fields fail to satisfy the same commutation relations, and he has put forward the conjecture that there might not exist wave equations for spins higher than unity that are free of all inconsistencies. It will, however, become apparent that this assessment is too pessimistic.

In recent years the interest in relativistic wave equations has mainly been focused on the instabilities mentioned above, yet again only some very special examples of wave equations have been considered. We believe the time has come for a detailed study of more general equations (1.1), not only for a better physical understanding of these difficulties, but also, in the first place, for possible practical applications. From a pragmatic point of view these acausalities and instabilities are quite irrelevant for the investigation of the electromagnetic and weak structure of the particles described by such wave equations. All actual calculations of form factors, polarizabilities, structure functions, etc. can be based on standard perturbation theory, i.e., on the Dyson expansion of the S matrix, and for this purpose it is sufficient to develop a consistent free field theory based on such wave equations. Of course it may very well be that none of these more general wave equations has anything to do with the strongly interacting particles as observed in nature; in that case, however, we also would be much less perturbed by their pathological behavior, and we could write off these equations as being of mere academic interest.

Here in this paper we want to investigate a large class of wave equations that appears to be most promising from both the theoretical and the practical points of view. We cannot expect every invariant wave equation (1.1) to be of interest in physics, and to prevent our investigation from degenerating into mere mathematics we have to impose certain restrictions on the class of wave equations to be considered. In view of the successful description of leptons offered by the Dirac equation, it is natural to look for a generalization of this theory. The Dirac equation is based on the following assumptions:

I. The state of the system is described by a multicomponent wave function which transforms locally under the Poincaré group, equation (1.2).

II. The free wave function obeys an invariant wave equation of first order, equation (1.1).

III. The wave equation describes one single particle (and its corresponding antiparticle); the solutions belonging to this single mass value $p^2 = m^2$ form a complete set.

We do not want to give up postulates I and II. Considering the experimentally observed multiplet structure of hadrons and their excited states we are led to replace the last requirement by the following:

III′. The wave equation describes a whole spectrum of particles (and their corresponding antiparticles); the solutions belonging to the various mass values $p^2 = m_\alpha^2 \geqslant 0$ form a complete set. We further demand the masses m_α to be uniquely determined by the wave equation, i.e., by the matrices β^μ and κ.

A wave equation is called "regular" if it satisfies the postulates I, II, and III′ above. To us, this class of regular wave equations appears to be the natural starting point for the investigation of the most general wave equation (1.1): It is large enough to contain interesting physics yet free of unnecessary complications; regular wave equations will be shown to be quite remarkably well behaved indeed. Only very few of the wave equations considered thus far are regular, especially none of the equations proposed for single particles of higher spin. It is characteristic of the wave equations (1.1) to describe more than one particle, and for all of these particles to be coupled by an interaction. In view of the observed multiplets of hadrons we believe that also the full mass spectra of such wave equations have to be taken seriously, and that we should not try to project out just one particular state. In our opinion the problem of a single particle with arbitrary spin is a purely academic one, as such particles do not exist in nature; all observed particles with higher spin are members of multiplets containing several particles with various masses and spins.

At first we shall restrict ourselves to wave functions with only a finite number of components. This is a more technical assumption, the theory of the finite-dimensional representations of the Lorentz group being much simpler than that for infinite dimensions. Besides, infinite-component wave equations seem to pose some problems of their own, as witnessed by the invalidity of the spin statistics and CPT theorems for infinite-component fields (Bogolubov, Logunov, and Todorov, 1975).

We can easily derive the mathematical conditions for a wave equation to be regular. The momentum space wave functions $w(\mathbf{p},\alpha\epsilon)$ corresponding to particles with momentum $\mathbf{p}$ satisfy, together with their appropriately defined adjoint spinors $\bar{w}(\mathbf{p},\alpha\epsilon)$, the usual orthogonality relations

$$\bar{w}(\epsilon\mathbf{p},\alpha\epsilon)\beta^0 w(\epsilon'\mathbf{p},\alpha'\epsilon')=\epsilon N_\alpha(\mathbf{p})\delta(\alpha,\alpha')\delta_{\epsilon\epsilon'} \tag{1.4a}$$

Here α stands for a set of indices that label the various solutions (see Section 3 below), and $\epsilon=\pm 1$ distinguishes the positive- and negative-frequency solutions. These orthogonality relations follow partly from the wave equation, and partly from the transformation properties of the spinors under the little-groups $SU(2)$ or $E(2)$. For a regular wave equation the norm $N_\alpha(\mathbf{p})$ is always different from zero: Assuming $N=0$ for a particular solution, the vector $\beta^0 w$ would then be orthogonal to all $\bar{w}(\epsilon\mathbf{p},\alpha\epsilon)$ that form a complete set. This would imply that $\beta^0 w=0$; hence the energy and also the mass of this state would then simply be free parameters and would not be uniquely determined by the wave equation, as originally required. We denote by $\bar{\mathfrak{W}}(\mathbf{p})$ the matrix having as rows all the spinors $\bar{w}(\epsilon\mathbf{p},\alpha\epsilon)$, and by $\mathfrak{W}(\mathbf{p})$ the matrix consisting of all column vectors $w(\epsilon\mathbf{p},\alpha\epsilon)$. These matrices are nonsingular as the spinors w and $\bar{w}$ form linearly independent sets. Equation (1.4a) can be written in matrix form as

$$\bar{\mathfrak{W}}(\mathbf{p})\beta^0\mathfrak{W}(\mathbf{p})=\mathrm{diag}[\pm N_\alpha(\mathbf{p})],\ N_\alpha\neq 0 \tag{1.4b}$$

Hence β^0 nonsingular is a necessary condition for a wave equation to be regular.

Here we want to consider only those regular wave equations where all the masses m_α are different from zero. For questions related to gauge invariance massless particles require special attention, and they will be discussed separately. In the rest frame of the particles we obtain from

(1.4b) with the help of the wave equation ($\mathbf{p}=0$)

$$\overline{\mathcal{W}}\kappa\mathcal{W}=\mathrm{diag}(\pm m_\alpha N_\alpha) \tag{1.4c}$$

We deduce that $\det\kappa\neq 0$ is a necessary (and sufficient) condition for a regular wave equation to contain only massive particles. In this case we can multiply the wave equation by κ^{-1} and thus assume with no loss of generality that $\kappa=\mathbf{1}$. [We observe that κ commutes with $D(g)$; therefore with β^μ also $\kappa^{-1}\beta^\mu$ transform like a vector.]

Summing up, in this paper we want to study all finite-component wave equations of the type

$$(-i\beta^\mu\partial_\mu+\mathbf{1})\psi(x)=0 \tag{1.5}$$

with nonsingular β^0. The mass spectrum of the wave equation is now determined by the inverse eigenvalues of β^0 and assumed to be real; with the unit matrix as the constant term in the wave equation we are explicitly excluding massless particles. The wave function and the β matrices transform under the Lorentz group according to (1.2) and (1.3a). The basic theoretical entities $\psi(x)$, β^μ, and $D(g)$ are only determined up to a common similarity transformation. We shall use this freedom to bring the transformation law $D(g)$ into the standard form described in Section 2 below. This implies that in general we cannot impose any further conditions on the matrices β^μ. In particular, we shall assume neither that β^0 is Hermitian nor that there exists a hermitizing matrix η such that $(\beta^\mu)^\dagger=\eta\beta^\mu\eta^{-1}$, as is customarily done. [From the definition of regular wave equations it follows that β^0 is equivalent to a real diagonal matrix, and hence also to $(\beta^0)^\dagger$. This, however, is a weaker assumption, which in general does not entail the existence of a hermitizing matrix. For that also $D(g^{-1})^\dagger=\eta D(g)\eta^{-1}$ is necessary, but here we shall not require the transformation law $D(g)$ to be pseudounitary.] Similarly, at first we shall not assume the wave equation to be manifestly covariant under parity, time reversal, or charge conjugation, which would impose additional limitations on the β^μ and $D(g)$. We believe the general formalism of relativistic wave equations should be developed independently of these discrete transformations (which are not even exact symmetries). From the theoretical point of view there is no actual need for any further restrictions, and we shall see that wave equations with quite general β^μ and $D(g)$ can be treated in a natural way. This not only has the procedural advantage that we can investigate a larger class of wave equations, but also, and perhaps even more importantly, in this way we avoid confounding the parity matrix with the metric operator in the definition of the adjoint wave function and the scalar product.

In Section 2 we describe a convenient realization of the matrices β^μ by means of Clebsch–Gordan coefficients of the rotation group. As we do not impose any particular algebra on the β^μ such an explicit expression is essential for all practical calculations. We evaluate the matrix elements of the β^μ between spin projectors in terms of Racah (6*j*) and 9*j* coefficients, using graphical techniques developed for angular momentum analysis. (These techniques are briefly summarized in Appendix A; in Appendix B we discuss a recursion formula for 9*j* symbols, correcting some unfortunate errors in the existing literature, and in Appendix C we list two particular Racah coefficients.) We further show that for every wave equation (1.1) there exists a matrix β_5 that anticommutes with all the β^μ and thus establishes a connection between particle and antiparticle solutions, even in theories without manifest *C* invariance. In Section 3 we construct a complete set of plane wave solutions and we describe a practical method for the determination of the mass spectrum. We define in Section 4 the adjoint wave function $\bar{\psi}(x)$ needed for a conserved and Lorentz invariant scalar product with the usual properties; in contrast to standard procedure our $\bar{\psi}(x)$ is not simply ("locally") related to $\psi^\dagger(x)$. In Section 5, starting from elementary matrix algebra, we derive for any regular wave equation (1.5) its Klein–Gordon divisor, of which we give three different expressions. By means of this Klein–Gordon divisor all the invariant functions associated with a given regular wave equation can be simply related to the corresponding invariant functions belonging to a set of Klein–Gordon equations (Section 6). In particular, we present a much simplified proof that the so-called contact terms vanish identically for all regular wave equations. Section 7 is devoted to the discrete transformations *P*, *T*, and *C*. In Section 8 we discuss the quantum field theories based on regular wave equations. Whereas, of course, it is possible to construct a free quantum field theory for almost any free wave equation we emphasize in Section 8 and in the following discussion (Section 9) the remarkably well-behaved nature of field theories based on regular wave equations. Although here we are only considering noninteracting theories, it will become apparent that regular wave equations are free of the worst instabilities and seem to offer the basis for consistent field theories of (multiplets of) particles with any spin.

2. THE MATRICES β^μ AND β_5

In the case of the Dirac equation all calculations can be performed without a specific realization of the Dirac matrices, utilizing only their simple algebraic properties. There have been several attempts to obtain new wave equations by suitable generalizations of the Dirac algebra.[2] We do not follow this purely algebraic approach: By postulating a certain algebra for the matrices β^μ only a small class of wave equations is selected, whereas we are here interested in a general and unbiased overall survey. At present there seems to be no fundamental reason for preferring one particular algebra to another; as higher dynamical symmetries appear to be only approximately realized in nature it is not evident why the matrices β^μ should have a simple algebraic structure at all. Besides, even such a simple generalization of the Dirac algebra as the DKP algebra (Fischbach, Louck, Nieto, and Scott, 1974b) is already quite intricate and makes practical calculations difficult. A convenient realization of the matrices β^μ is thus essential for any explicit calculations with more general wave equations (1.1).

For the systematic classification and study of relativistic wave equations we consider the "type" of the wave function to be given, i.e., its transformation law $D(g)$ under the Lorentz group. The matrices β^μ corresponding to a given $D(g)$ are then determined from their postulated vector character, $D^{-1}\beta^\mu D = L^\mu{}_\nu \beta^\nu$. The explicit expression for the β^μ is thus closely related to an appropriate standardization of the transformation law $D(g)$. Every finite-dimensional representation of the Lorentz group is completely reducible (van der Waerden, 1932), and we may assume $D(g)$ to be already given in completely reduced form, $D(g)=\Sigma\oplus D^i(g)$, where the index $i=1,2,\ldots,N$ denotes the various irreducible representation contained in $D(g)$. (The same irreducible representation may occur more than once in this decomposition; thus it may be that $D^i = D^k$ for $i \neq k$). For the further standardization of the $D^i(g)$ we observe that all inequivalent, irreducible, finite-dimensional representations of the Lorentz group can be written as the direct product of two rotation matrices, $D^{AB}(g) = D^A(g)\oplus D^B(g)^{\dagger-1}$, A and B being any integers or half-integers, the dagger denoting the Hermitian conjugate matrix. Here $D^A(g)$ is the usual rotation matrix for angular momentum A [analytically continued from $SU(2)$ to $SL(2,C)$; for the relevant facts about the representations of $SL(2,C)$ see, e.g., Wightman (1960)]. This standardization of the irreducible representations of $SL(2,C)$ has the advantage that the decoupling coefficients for the Lorentz group are now simply products of two ordinary Clebsch–Gordan coefficients of the rotation group.

Corresponding to this completely reduced form of $D(g)$ we denote the rows and columns of β^μ according to the irreducible components i,k of $D(g)$, each of which is characterized by a pair of angular momenta, $i=(A_i,B_i)$, $k=(A_k,B_k)$. We subdivide the matrix β^μ into blocks of $(2A_i+1)(2B_i+1)$- by $(2A_k+1)(2B_k+1)$-dimensional rectangular matrices (β^μ_{ik});

[2]See, e.g., Chapter V of Corson (1953).

the elements within such a block are labeled by a pair of indices $a_i b_i$, with $-A_i \leqslant a_i \leqslant A_i$, $-B_i \leqslant b_i \leqslant B_i$, and a similar pair of column indices $a_k b_k$. Using Schur's lemma, we immediately deduce from the required vector character of the β^μ, $D^{-1}\beta^\mu D = L^\mu{}_\nu \beta^\nu$, that the block of matrix elements (β^μ_{ik}) connecting the ith with the kth irreducible component of $D(g)$ is given, up to an arbitrary numerical factor, by a decoupling coefficient of the Lorentz group, that is, by a product of two Clebsch–Gordan coefficients of the rotation group (Lyubarskii, 1960):

$$(\beta^\mu_{ik})_{a_i b_i; a_k b_k} = -b_{ik}\left[(2A_i+1)(2B_k+1)\right]^{-1/2}(\sigma^\mu)_{\tau\tau'} \langle A_i a_i | \tfrac{1}{2}\tau A_k a_k\rangle \langle B_i b_i \tfrac{1}{2}\tau' | B_k b_k\rangle \tag{2.1}$$

Here the b_{ik} are arbitrary complex parameters, and we have chosen the sign and the other numerical factors to obtain a simple graphical representation of these matrices (see Figure 1). This choice entails, among other things, a simple expression for the reduced mass matrix $\Lambda(s)$, equations (2.4) and (2.5) below. The $\sigma^\mu = (\mathbf{1}, \boldsymbol{\sigma})$ are the usual Pauli matrices which transform between Cartesian and spherical vector components, and there is a summation over the double indices τ and τ' with values $\pm 1/2$. With this realization of the β^μ all the algebraic problems encountered in the study of the general wave equation (1.1) are now reduced to problems of angular momentum analysis; for these powerful and elegant graphical techniques have been developed (Yutsis, Levinson, and Vanagas, 1962; El-Baz and Castel, 1972) which make actual calculations with more general wave equations manageable.

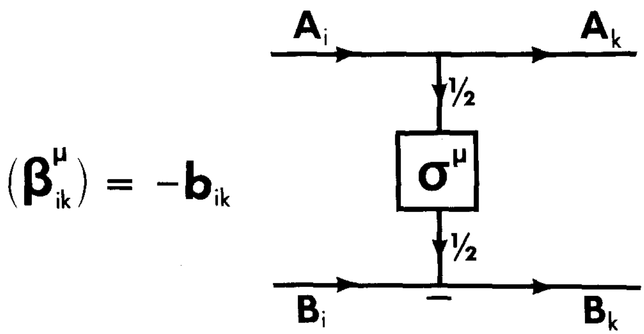


Fig. 1. Graphical representation for the block of matrix elements (β^μ_{ik}). As explained in Appendix A, each vertex corresponds to a Clebsch–Gordan coefficient.

We observe that the transformation law $D(g)$ necessarily has to be reducible. For a vector β^μ to exist, $D(g)$ has to contain at least one pair of "interlocking" (or "linked") irreducible components i,k satisfying $A_i = A_k \pm 1/2$, and, independently, $B_i = B_k \pm 1/2$, i.e., the representation $i=(A_i, B_i)$ is contained in the direct product $(A_k, B_k)\otimes(\frac{1}{2},\frac{1}{2})$. The only nonvanishing matrix elements of β^μ are those blocks (β^μ_{ik}) connecting two interlocking representations i and k. In Figures 2a–2f we present some examples of the linkage diagrams of the most common wave equations. There each point in the (A,B) plane represents a certain irreducible component $D^i(g)$, and a line joining two such points corresponds to a nonvanishing linkage parameter b_{ik}. For a given transformation law $D(g)$ the matrices β^μ are

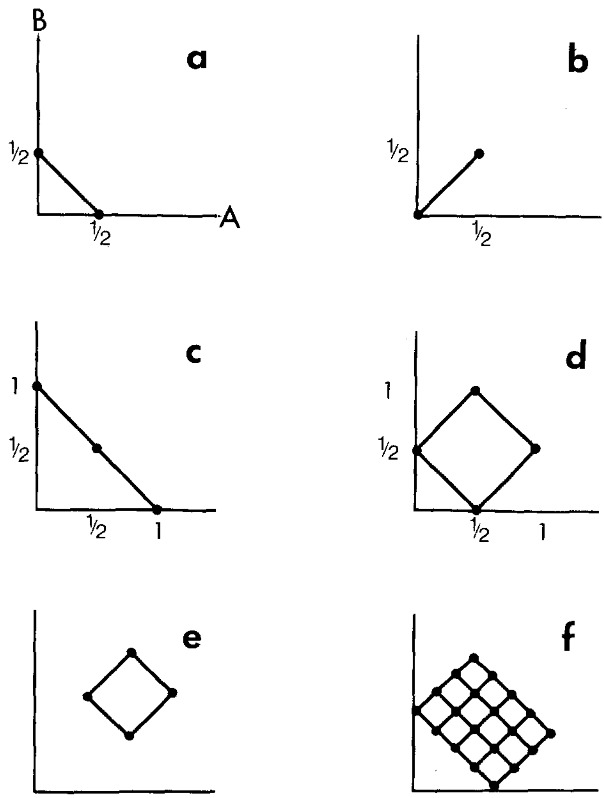


Fig. 2. Linkage diagrams for some common wave equations: (a) Dirac equation. (b) Duffin, Kemmer, Petiau spin 0. (c) Proca equation; Duffin, Kemmer, Petiau spin 1. (d) Fierz–Pauli spin 3/2. (e) Rarita–Schwinger equation corresponding to the direct product of the Dirac representation with the representation $(n/2, n/2)$. (f) Bhabha equation; there the irreducible representations fill a rectangle of any size, depending on the particular representation of $so(5)$.

not uniquely determined: There is a free parameter b_{ik} for each interlocking pair i,k of irreducible representations, reflecting some flexibility in the masses (and parities) of the states which can be accommodated by a given type of wave function. Postulating a certain algebra for the β^μ not only restricts the type of the wave function, i.e., its transformation law $D(g)$, but also implies, for better or worse, a definite choice of the parameters b_{ik} and hence a uniquely determined mass spectrum. Unfortunately most of the wave equations studied by algebraic techniques suffer from a rather unphysical mass spectrum, like decreasing mass values for increasing spin. Here we do not assume the matrices β^μ to obey any particular algebra and we treat the b_{ik} as arbitrary free parameters (which in principle can be determined from a given mass spectrum).

Equation (2.1) and Figure 1 express the vector character of the matrices β^μ under Lorentz transformations. A physically more useful realization of the β^μ is obtained by recoupling certain angular momenta in Figure 1. This is most easily done in the case of the matrix β^0 (see Figure 3). Introducing the completeness relation of Appendix A, equation (A.8), for the A_kB_k lines, we can then contract the ensuing triangle of internal lines to a point, obtaining a Racah coefficient according to equation (A.9). We find it convenient to introduce the spin projectors $\chi_i(s\sigma)$, a set of $(2A_i+1)(2B_i+1)$-dimensional column vectors; their components are labeled by the pair of indices a_ib_i and are given by the Clebsch–Gordan coefficient

$$[\chi_i(s\sigma)]_{a_ib_i} = \langle A_ia_iB_ib_i|s\sigma\rangle \tag{2.2a}$$

Similarly we define the corresponding adjoint row vectors $\chi_i^\dagger(s\sigma)$

$$[\chi_i^\dagger(s\sigma)]_{a_ib_i} = \langle s\sigma|A_ia_iB_ib_i\rangle \tag{2.2b}$$

These spin projectors $\chi_i(s\sigma)$ form a complete and orthonormal set of vectors in the $(2A_i+1)(2B_i+1)$-dimensional space associated with the irreducible representation $D^i(g)$, $i=(A_i,B_i)$:

$$\sum_{s\sigma} \chi_i(s\sigma)\otimes\chi_i^\dagger(s\sigma) = \mathbf{1}_i \tag{2.3a}$$

$$\chi_i^\dagger(s\sigma)\chi_i(s'\sigma') = \delta_{ss'}\delta_{\sigma\sigma'} \tag{2.3b}$$

i fixed, and where $\mathbf{1}_i$ denotes the unit matrix in that space.

In terms of these spin projectors we can now write for the blocks of matrix elements of β^0

$$(\beta_{ik}^0) = \sum_{s\sigma} \Lambda_{ik}(s)\chi_i(s\sigma)\otimes\chi_k^\dagger(s\sigma) \tag{2.4}$$

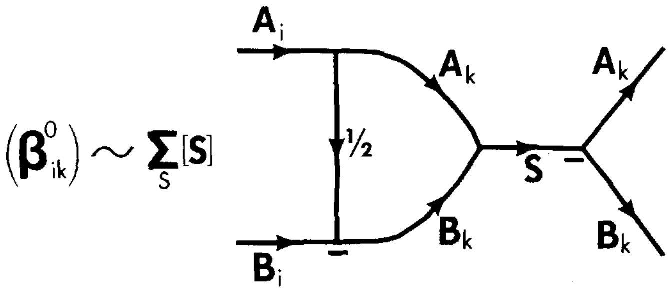


Fig. 3. Recoupling of the $A_k B_k$ lines of (β^0_{ik}) with the help of the completeness relation (A.8) defined in Appendix A.

where we have suppressed the magnetic quantum numbers $a_i b_i; a_k b_k$. The coefficients $\Lambda_{ik}(s)$ are proportional to a Racah coefficient,

$$\Lambda_{ik}(s) = b_{ik}(-1)^{\Phi} \begin{Bmatrix} A_i & B_i & s \\ B_k & A_k & \frac{1}{2} \end{Bmatrix} \tag{2.5}$$

with the phase $\Phi = A_i + B_k + s + \frac{1}{2}$ [the overall phase of $\Lambda_{ik}(s)$ is actually independent of s]; the b_{ik} are the free parameters contained in the matrices β^{μ} according to (2.1). Using the orthogonality and completeness relations of the spin projectors we get

$$(\beta^0_{ik})\chi_k(s\sigma) = \Lambda_{ik}(s)\chi_i(s\sigma) \tag{2.6a}$$

$$\chi_i^{\dagger}(s\sigma)(\beta^0_{ik}) = \Lambda_{ik}(s)\chi_k^{\dagger}(s\sigma) \tag{2.6b}$$

We note that the last equation follows without any hermiticity requirements on the matrix β^0. We shall find in Section 3 that the spin projectors $\chi_i(s)$ are also the fundamental building blocks for the spinors describing particles of spin s at rest, and that the mass spectrum of the wave equation can be simply determined from the $\Lambda_{ik}(s)$ [see (3.20) below].

The evaluation of the general matrix elements $\chi_i^{\dagger}(s\sigma)(\beta^{\mu}_{ik})\chi_k(s'\sigma')$, sketched in Figure 4, is only slightly more involved. We introduce the completeness relation for the internal $j = \frac{1}{2}$ lines; after a slight rearrangement we obtain the graph shown in Figure 5. According to Appendix A, equation (A.10), the matrix elements can then be immediately identified with a $9j$ symbol times a Clebsch–Gordan coefficient. Following the convention of Schwinger (1952) for the reduced matrix elements we find (i,k fixed; we suppress a sum over the magnetic quantum numbers $a_i b_i$ and $a_k b_k$):

$$\chi_i^{\dagger}(s\sigma)(\beta^{\mu}_{ik})\chi_k(s'\sigma') = \delta_0^{\mu}\Lambda_{ik}^{(0)}(s,s')[s]^{-1/2}\delta_{ss'}\delta_{\sigma\sigma'} + (1-\delta_0^{\mu})\Lambda_{ik}^{(1)}(s,s')[s']^{-1/2}\langle s\sigma 1\mu|s'\sigma'\rangle \tag{2.7}$$

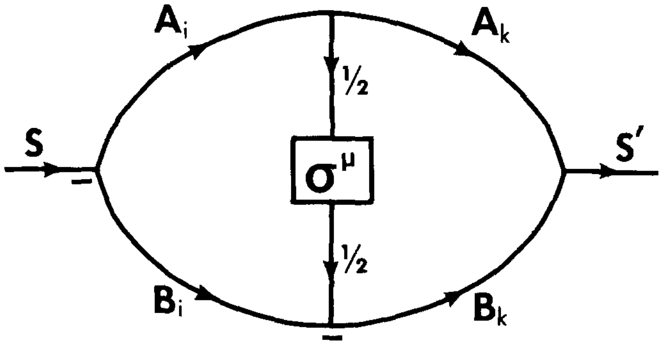


Fig. 4. Graphical expression for the matrix elements of β^μ between spin projectors, equation (2.7).

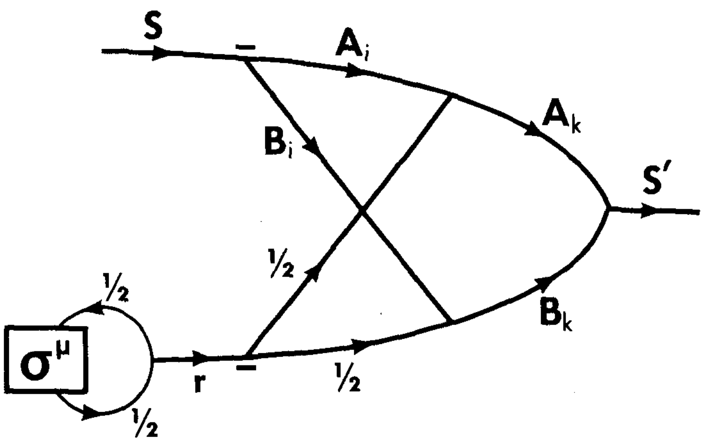


Fig. 5. After rearranging some lines in Figure 4 and using the completeness relation (A.8) for the $j=1/2$ lines, the ensuing graph is immediately identified with a $9j$ symbol, as defined in (A.10) of Appendix A.

with the reduced matrix elements being proportional to a $9j$ symbol,

$$\Lambda_{ik}^{(r)}(s,s')=b_{ik}\left[\tfrac{1}{2},r,s,s'\right]^{1/2}\begin{Bmatrix} A_i & \frac{1}{2} & A_k \\ B_i & \frac{1}{2} & B_k \\ s & r & s' \end{Bmatrix} \tag{2.8}$$

$r=0,1$. There we have introduced the abbreviation $[j,l,s,\ldots]=(2j+1)(2l+1)(2s+1)\cdots$. In the "mixed" Clebsch–Gordan coefficient of (2.7) μ denotes the Cartesian vector indices, i.e., $\langle s\sigma 1\mu|s'\sigma'\rangle=\langle s\sigma 1\rho|s'\sigma'\rangle V_{\rho\mu}$, where

V is the usual similarity transformation connecting Cartesian and spherical vector indices ($\mu = 1,2,3$; $\rho = \pm 1,0$). For $r=0$ we have

$$\Lambda_{ik}^{(0)}(s,s') = \delta_{ss'}[s]^{1/2}\Lambda_{ik}(s) \tag{2.9}$$

Further useful relations can be derived from the commutation relations of the β^τ with the generators $\Omega^{\mu\nu}$ of the Lorentz group, $D(g) = \exp[-(i/2)\omega\cdot\Omega]$,

$$i[\Omega^{\mu\nu}, \beta^\tau] = G^{\mu\tau}\beta^\nu - G^{\nu\tau}\beta^\mu \tag{2.10}$$

characteristic of any 4-vector; G is the metric tensor $(1,-1,-1,-1)$. In particular we obtain for the generators $\mathbf{J} = (\Omega^{23}, \Omega^{31}, \Omega^{12})$ of rotations, and $\mathbf{K} = (\Omega^{01}, \Omega^{02}, \Omega^{03})$ of boosts

$$[\mathbf{J}, \beta^0] = 0 \tag{2.11a}$$

$$i[\mathbf{K}, \beta^0] = \boldsymbol{\beta} \tag{2.11b}$$

$$[J^p, \beta^q] = i\epsilon_{pqr}\beta^r \tag{2.11c}$$

$$i[K^p, \beta^q] = \delta_{pq}\beta^0 \tag{2.11d}$$

Not all of these commutation relations are independent of each other. Starting from a matrix β^0 satisfying (2.11a), the second equation (2.11b) may serve as the definition of the corresponding matrices $\boldsymbol{\beta}$. Of the other equations we only have to satisfy (2.11d) for one particular nonvanishing commutator, say for $p=q=3$. All the remaining commutation relations then simply follow from a repeated application of the Jacobi identity. Hence the necessary and sufficient condition for the matrix β^0, satisfying (2.11a), to generate a four-vector β^μ via (2.11b) is

$$-[K^3, [K^3, \beta^0]] = \beta^0 \tag{2.12}$$

This self-consistency condition for the matrix β^0 is easily understood: equation (2.12) together with (2.11b) are the obvious requirements for $D^{-1}[b_3(\zeta)]\beta^0 D[b_3(\zeta)] = \beta^0 \cosh\zeta + \beta^3 \sinh\zeta$ to hold.

In our standardization of $D(g)$ and its irreducible components the generators $\mathbf{J}$ and $\mathbf{K}$ have a simple expression. For the irreducible representation $D^{AB}(g)$ we find

$$\mathbf{J}^{AB} = \mathbf{J}^A \otimes \mathbf{1}^B + \mathbf{1}^A \otimes \mathbf{J}^B \tag{2.13a}$$

$$i\mathbf{K}^{AB} = \mathbf{J}^A \otimes \mathbf{1}^B - \mathbf{1}^A \otimes \mathbf{J}^B \tag{2.13b}$$

where $\mathbf{J}^A$ are the standard angular momentum matrices belonging to angular momentum A. In the following we shall simply write (**A**) for $\mathbf{J}^A \otimes \mathbf{1}^B$, and similarly (**B**) for $\mathbf{1}^A \otimes \mathbf{J}^B$. From the above equations we obtain various relations between the β^μ and angular momentum matrices. From (2.11b) we deduce for the blocks of matrix elements of the $\boldsymbol{\beta}$

$$(\boldsymbol{\beta}_{ik}) = 2\left[(\mathbf{A})(\beta_{ik}^0) - (\beta_{ik}^0)(\mathbf{A})\right] \tag{2.14a}$$

or

$$(\boldsymbol{\beta}_{ik}) = -2\left[(\mathbf{B})(\beta_{ik}^0) - (\beta_{ik}^0)(\mathbf{B})\right] \tag{2.14b}$$

To further simplify the notation we have dropped the indices i,k on the angular momentum matrices; the irreducible representations to which these matrices belong are uniquely determined by the order in which the various matrices occur. Thus, an angular momentum matrix to the left of the block (β_{ik}^0) belongs to the representation D^i; for example, $(\mathbf{A})(\beta_{ik}^0)$ stands for the product $(\mathbf{J}^{A_i} \otimes \mathbf{1}^{B_i})(\beta_{ik}^0)$, and similarly $(\beta_{ik}^0)(\mathbf{B})$ means $(\beta_{ik}^0)(\mathbf{1}^{A_k} \otimes \mathbf{J}^{B_k})$. The above relations trivially follow from our graphical representation of the matrices β^μ as given in Figure 1. We only have to use that $\boldsymbol{\sigma} = 2\mathbf{J}^{(1/2)}$, and the simple rules for the shifting of angular momentum matrices along the lines of a Clebsch–Gordan coefficient, as derived in Appendix A, equation (A.5).

Taking the matrix elements of (2.14a) between the spin projectors $\chi_i^\dagger(s)$ and $\chi_k(s')$, we arrive at a relation between the corresponding reduced matrix elements:

$$\Lambda_{ik}^{(1)}(s,s') = 2\left[A_i(s,s')\Lambda_{ik}(s') - \Lambda_{ik}(s)A_k(s,s')\right] \tag{2.15}$$

The $A_i(s,s')$ are the reduced matrix elements of the angular momentum matrices $(\mathbf{A}_i)$ in the basis of the spin projectors,

$$\chi_i^\dagger(s\sigma)(A_i^q)\chi_i(s'\sigma') = A_i(s,s')\left[s'\right]^{-1/2}\langle s\sigma 1 q | s'\sigma'\rangle \tag{2.16a}$$

$$A_i(s,s') = (-1)^{A_i+B_i+s}\left[s,s'\right]^{1/2}(A_i\|J\|A_i)\begin{Bmatrix} s & 1 & s' \\ A_i & B_i & A_i \end{Bmatrix} \tag{2.16b}$$

with the usual $(A\|\mathbf{J}\|A) = [A(A+1)(2A+1)]^{1/2}$. Equation (2.15) constitutes a recursion formula relating the $9j$ symbols occurring in (2.8) with $r=1$ to the corresponding $9j$ symbols with $r=0$, i.e., Racah coefficients. As this appears to be a new type of recursion relation, and as there are some unfortunate errors in the tables of $9j$ symbols of even recently published books, we discuss this and similar relations in more detail in Appendix B.

From the commutators of the β^q with the generators K^p, $p,q=1,2,3$, equation (2.11d), we derive that

$$\delta_{pq}(\beta^0_{ik})+i\epsilon_{pqr}(\beta^r_{ik})=2[(A^p)(\beta^q_{ik})-(\beta^q_{ik})(A^p)] \tag{2.17a}$$

This equation is the direct generalization of the well-known relation $\sigma_p\sigma_q=\delta_{pq}+i\epsilon_{pqr}\sigma_r$ satisfied by the Pauli matrices. In fact, (2.17a) at once follows from this property of the $\boldsymbol{\sigma}$ and the identity expressed in equation (A.5) for the angular momentum matrices. For $p=q$ we obtain, similar to (2.14a),

$$(\beta^0_{ik})=2\{(A^p)(\beta^p_{ik})-(\beta^p_{ik})(A^p)\} \tag{2.17b}$$

p not summed. From (2.14a) and (2.17b) [or directly from (2.12)] we deduce the identity (p fixed)

$$(\beta^0_{ik})=4[(A^p)^2(\beta^0_{ik})+(\beta^0_{ik})(A^p)^2-2(A^p)(\beta^0_{ik})(A^p)] \tag{2.18}$$

which is the necessary and sufficient condition for generating a 4-vector β^μ from the matrix β^0. This last equation entails a sum rule for the Racah coefficients appearing in the reduced matrix elements $\Lambda_{ik}(s)$, equation (2.5), which here we do not need to spell out in more detail.

With our explicit realization of the matrices β^μ it is now straightforward to compute their various products and commutators. We do not see much point in studying the algebra generated by the β^μ; simple algebraic relations will only hold for some particular $D(g)$, and even then only for special values of the linkage parameters b_{ik}. For all practical purposes we found the graphical expression of the β^μ more convenient than some ad hoc algebra postulated for the β^μ.

The analog of the Dirac matrix γ_5 has a particularly simple expression in our realization of the β^μ. We define the diagonal matrix $\beta_5(=\beta^5)$ in terms of its blocks of matrix elements (β^5_{ik}) as

$$(\beta^5_{ik})=(-1)^{2A_i}\delta_{ik}\mathbf{1}_i \tag{2.19}$$

where $\mathbf{1}_i$ denotes the $(2A_i+1)(2B_i+1)$-dimensional unit matrix. [In passing we note the difference between the symbols δ_{ik} and $\delta(A_i,A_k)\delta(B_i,B_k)$.] β_5 is Hermitian and unitary as

$$(\beta_5)^2=\mathbf{1} \tag{2.20}$$

β_5 commutes with all Lorentz transformations,

$$\beta_5 D(g)=D(g)\beta_5 \tag{2.21}$$

whereas it anticommutes with all the β^μ,

$$\beta_5 \beta^\mu = -\beta^\mu \beta_5 \tag{2.22}$$

$(\beta_5 \beta^\mu \beta_5)_{ik} = (-1)^{2A_i+2A_k}(\beta^\mu_{ik}) = -(\beta^\mu_{ik})$ as the only nonvanishing matrix elements of (β^μ_{ik}) are those where i and k are interlocked, i.e., where $A_i = A_k \pm \frac{1}{2}$.

Evidently this matrix β_5 always exists without any further restrictions on the β^μ or the transformation law $D(g)$. By means of β_5 we will obtain a direct connection between the particle and antiparticle solutions of the general wave equation, even without postulating the wave equation to be manifestly covariant under charge conjugation.

3. MASS SPECTRUM, COMPLETE SET OF PLANE WAVE SOLUTIONS

Before we begin with the actual construction of a complete set of solutions of the free wave equation (1.5), we want to briefly review the kinematical aspects of free particles with arbitrary spin in a manner that does not depend on the particular choice of the wave function, utilizing only general invariance arguments. This is done not only to introduce our notation and normalization conventions, but, primarily, since by such techniques we will construct and interpret the various solutions of the free wave equation.

The quantum mechanical symmetry operators that correspond to the elements of the restricted Poincaré group (no reflections) can be chosen (Wigner, 1939), within each coherent subspace, to form a unitary representation of $ISL(2,C)$. We label the elements of this group by (a,g), where a denotes the space–time translation by the 4-vector a^μ, and g is an element of $SL(2,C)$. The symmetry operators are normalized to satisfy the group property

$$U(a_1,g_1)\,U(a_2,g_2) = U(a_1 + L_1 a_2, g_1 g_2) \tag{3.1}$$

Here L_1 is the Lorentz transformation corresponding to the elements g_1 of $SL(2,C)$; we write for this homomorphism

$$L(g)^\mu{}_\nu = \tfrac{1}{2}\mathrm{Tr}\left(\tilde{\sigma}^\mu g \sigma_\nu g^\dagger\right) \tag{3.2}$$

the dagger denoting the Hermitian conjugate. The σ^μ are again the usual set of Pauli matrices, and $\tilde{\sigma}^\mu = P^\mu{}_\nu \sigma^\nu = (1, -\boldsymbol{\sigma})$, $P^\mu{}_\nu$ being the parity matrix. The $U(a,g)$ relate the quantum mechanical descriptions of different inertial frames, or, in the active interpretation, describe the behavior of the system under active Lorentz transformations; thus all the information about the relativistic kinematics of free particles with any spin is already contained in (3.1).

By definition, the state vectors $|s\sigma\rangle$ which describe a massive particle of spin s at rest transform under rotations $r \in SU(2)$ according to

$$U(r)|s\sigma\rangle = |s\sigma'\rangle D^s(r)_{\sigma'\sigma} \tag{3.3}$$

with $D^s(r)$ the usual rotation matrix, and a sum over double indices. We obtain the states $|ps\sigma\rangle$ for the particle in motion by applying active Lorentz transformations to the states at rest. We define the standard states $|ps\sigma\rangle$ corresponding to a particle with energy–momentum $p=(E,\mathbf{p})$ as

$$|ps\sigma\rangle = U([p])|s\sigma\rangle \tag{3.4}$$

Here $[p] \in SL(2,C)$ is a boost that brings the particle from rest to the 4-momentum p, i.e., for which $L([p])p_r = p$, $p_r = (m,0,0,0)$. From equations (3.1), (3.3), and (3.4) we obtain the well-known transformation law of the states $|ps\sigma\rangle$ under arbitrary Lorentz transformations

$$U(a)|ps\sigma\rangle = e^{ip\cdot a}|ps\sigma\rangle \tag{3.5a}$$

$$U(g)|ps\sigma\rangle = |p's\sigma'\rangle D^s[W(g,p)]_{\sigma'\sigma} \tag{3.5b}$$

with $p' = L(g)p$, and the Wigner rotation

$$W(g,p) = [p']^{-1}g[p] \tag{3.5c}$$

Equation (3.4) implies a relativistic normalization of the states which we chose to be

$$\langle ps\sigma|p's'\sigma'\rangle = 2E(\mathbf{p})\delta(\mathbf{p}-\mathbf{p}')\delta_{ss'}\delta_{\sigma\sigma'} \tag{3.5d}$$

$E(\mathbf{p}) = {}_+(\mathbf{p}^2+m^2)^{1/2}$. Due to the kinematic spin rotation (3.5c) the physical significance of the polarization index σ depends on the particular choice of the boost $[p]$, which is only determined up to an arbitrary rotation. The two most commonly used boosts are

$$[p]_s = r(p)b_3(\zeta)r^{-1}(p) \tag{3.6a}$$

and

$$[p]_h = r(p)b_3(\zeta) \tag{3.6b}$$

In these formulas $r(p)$ means a rotation that brings the positive z axis into the direction of the momentum $\mathbf{p}$; $b_3(\zeta)$ is a pure Lorentz transformation in the direction of the positive z axis with the rapidity ζ given by $\cosh\zeta = E/m$. For the boost $[p]_s$, σ has the physical significance of the z component of the spin, $\sigma = s_z$; we denote the corresponding "covariant spin states" by $|pss_z\rangle$. The boost $[p]_h$ generates the helicity states $|psh\rangle$, where $\sigma = h$ measures the component of the spin in the direction of motion.

The operator U_P representing space inversion can be chosen to be unitary (Wigner, 1965); within a coherent subspace it can be normalized to

$$U_P^2 = \mathbf{1} \tag{3.7a}$$

With the elements of the restricted Poincaré group the operator U_P satisfies the group multiplication

$$U_P^{-1}U(a,g)U_P = U(\tilde{a},\tilde{g}) \tag{3.7b}$$

The parity-transformed elements $(\tilde{a},\tilde{g})$ of $ISL(2,C)$ are given by

$$\tilde{a} = Pa = (a^0, -\mathbf{a}), \qquad \tilde{g} = g^{\dagger -1} \tag{3.7c}$$

where by definition $L(\tilde{g}) = PL(g)P^{-1}$.

It follows from (3.7b) that the states

$$U_P|\tilde{p}s\sigma'\rangle D^s\big([\tilde{p}]^\dagger[p]\big)_{\sigma'\sigma} \tag{3.8a}$$

$\tilde{p} = (E, -\mathbf{p})$, transform under the restricted Poincaré group in exactly the same way as the states $|ps\sigma\rangle$. [We note that $[\tilde{p}]^\dagger[p] \in SU(2)$.] Assuming no doubling of states we deduce that

$$U_P|ps\sigma\rangle = \eta_P|\tilde{p}s\sigma'\rangle D^s\big([\tilde{p}]^\dagger[p]\big)_{\sigma'\sigma} \tag{3.8b}$$

with the intrinsic parity $\eta_P = \pm 1$. In particular, we find for the covariant spin states that $[\tilde{p}]_s^\dagger[p]_s = \mathbf{1}$ and therefore

$$U_P|pss_z\rangle = \eta_P|\tilde{p}ss_z\rangle \tag{3.9}$$

We obtain for the helicity states

$$[\tilde{p}]_h^\dagger[p]_h = r^{-1}(\tilde{p})r(p) = r_3(\Phi)r_2^{-1}(\pi) \tag{3.10}$$

The so-defined rotation $r_3(\Phi)$ about the third axis depends on the further standardization of the rotation $r(p)$ which brings the z axis into the direction of the momentum $\mathbf{p}$, $r(p)$ being only determined up to a rotation around the z axis. Following Wigner (1959) we denote the matrices $D^s[r_2(\pi)]$ by C^s; they have the matrix elements

$$C^s_{\sigma\sigma'} = D^s[r_2(\pi)]_{\sigma\sigma'} = (-1)^{s+\sigma}\delta_{\sigma,-\sigma'} \tag{3.11a}$$

The importance of the matrices C^s stems from the fact that they relate the representation $D^s(r)$ of $SU(2)$ to its complex conjugate,

$$D^s(r)^* = C^s D^s(r)(C^s)^{-1}, \qquad r \in SU(2) \tag{3.11b}$$

The real matrices C^s are unitary and satisfy

$$(C^s)^2 = (-1)^{2s} \tag{3.11c}$$

For $s = 1/2$ we shall simply write C for $C^{1/2}$, i.e., $C = C^{1/2} = r_2(\pi) = -i\sigma_y$. Combining (3.8b), (3.10), and (3.11a) we find for the transformation law of the helicity states under parity

$$U_P|psh\rangle = \eta_P(-1)^{s+h}e^{ih\Phi}|\tilde{p}, s, -h\rangle \tag{3.12}$$

where the phase angle $\Phi = \Phi(p)$ is determined from (3.10).

Finally, time reversal can be represented by an antiunitary operator A_T (Wigner, 1965) which within a coherent subspace can be normalized to

$$A_T^2 = \epsilon_T \mathbf{1} \tag{3.13a}$$

$\epsilon_T = \pm 1$ being a real phase factor. The operator A_T satisfies the following group multiplication with the operators $U(a,g)$ of the restricted Poincaré group:

$$A_T^{-1}U(a,g)A_T = U(-\tilde{a},\tilde{g}) \tag{3.13b}$$

where again $\tilde{a} = (a^0, -\mathbf{a})$ and $\tilde{g} = g^{\dagger -1}$ [equation (3.7c)]. The states

$$A_T\{|\tilde{p}s\sigma'\rangle D^s([\tilde{p}]^\dagger[p]C)_{\sigma'\sigma}\} \tag{3.14a}$$

transform under the restricted Poincaré group in exactly the same way as the states $|ps\sigma\rangle$, the C matrix in (3.14a) reflecting the antiunitarity of A_T. Assuming no doubling of states we find

$$A_T|ps\sigma\rangle = \eta_T|\tilde{p}s\sigma'\rangle D^s([\tilde{p}]^\dagger[p]C)_{\sigma'\sigma} \tag{3.14b}$$

For the covariant spin and the helicity states we obtain the transformation laws

$$A_T|pss_z\rangle = \eta_T(-1)^{s-s_z}|\tilde{p},s,-s_z\rangle \tag{3.15a}$$

$$A_T|psh\rangle = \eta_T e^{-ih\Phi}|\tilde{p}sh\rangle \tag{3.15b}$$

the phase angle Φ being again determined from (3.10). Independently of the special choice of the standard states $|ps\sigma\rangle$ it directly follows from (3.14b) that for a particle of spin s

$$A_T^2 = (-1)^{2s}\mathbf{1} \tag{3.16}$$

The solutions of relativistic wave equations offer a concrete realization of the abstract state vectors considered above, and these general invariance arguments can be used quite literally to construct a complete set of solutions of the free wave equation (1.5): We first have to obtain the wave functions describing particles at rest, and then boost to arbitrary momentum. With our standardization of the wave equation (1.5) its mass spectrum is determined by the inverse eigenvalues of β^0; for a regular wave equation β^0 is assumed to have a complete set of eigenvectors belonging to real and nonvanishing eigenvalues. We call $u(\alpha)$ the eigenvectors to positive eigenvalues $\lambda_\alpha = 1/m_\alpha$ of β^0,

$$\beta^0 u(\alpha) = \lambda_\alpha u(\alpha), \qquad \lambda_\alpha > 0 \tag{3.17a}$$

Here α stands for a triplet of indices to distinguish the various eigenvectors, $\alpha = (s\sigma\rho)$: s and σ denote the spin and its z component, and ρ is a degeneracy parameter to label different states with the same spin. The matrix β_5 defined in (2.19) anticommutes with β^0; hence there exists to every $u(\alpha)$ a corresponding spinor $v(\alpha)$ belonging to the negative eigenvalue $-\lambda_\alpha$:

$$v(\alpha) = \beta_5 u(\alpha) \tag{3.17b}$$

$$\beta^0 v(\alpha) = -\lambda_\alpha v(\alpha) \tag{3.17c}$$

All the $u(\alpha)$ together with the $v(\alpha)$ form a complete set.

The actual determination of the mass spectrum of a given wave equation can be greatly simplified with the help of our explicit expression of the matrix β^0. By definition, to describe a particle of spin s at rest, the spinors $u(\alpha)$ have to transform under rotations $r \in SU(2)$ as required by (3.3),

$$D(r)u(s\sigma\rho) = u(s\sigma'\rho)D^s(r)_{\sigma'\sigma} \tag{3.18a}$$

where $D(r)$ is the given transformation law of the wave function, equation (1.2). In accordance with the assumed decomposition of $D(g)$ into a direct sum of irreducible representations $D^i(g)$, $i=1,2,\ldots,N$, and with arranging the matrix elements of β^μ into the blocks (β^μ_{ik}), we break up the spinors $u(\alpha)$ into $(2A_i+1)(2B_i+1)$-component column vectors $u_i(\alpha)$, labeling their components by the pair of indices $a_i b_i$, $-A_i \leqslant a_i \leqslant A_i$, $-B_i \leqslant b_i \leqslant B_i$. From (3.18a) these column vectors $u_i(\alpha)$ transform under rotations as

$$D^i(r)u_i(s\sigma\rho)=u_i(s\sigma'\rho)D^s(r)_{\sigma'\sigma} \tag{3.18b}$$

i not summed. Utilizing that for rotations $D^i(r)$ is simply the direct product of the two rotation matrices $D^{A_i}(r)$ and $D^{B_i}(r)$, we at once obtain from Schur's lemma that the components of $u_i(\alpha)$ are given, up to a constant factor, by the spin projectors $\chi_i(s\sigma)$ introduced in (2.2a):

$$u_i(s\sigma\rho)=x_i(s\rho)\chi_i(s\sigma) \tag{3.19}$$

The constants $x_i(s\rho)$ are determined by the eigenvalue equation $\beta^0 u(\alpha)=\lambda_\alpha u(\alpha)$. Combining (3.19) with (2.6a) for the effect of the (β^0_{ik}) on the spin projectors $\chi_k(s)$ we find that the eigenvalue problem of β^0 can be reduced, for each value of the spin s, to an eigenvalue problem of a corresponding N-dimensional matrix $\Lambda(s)$, N being the number of irreducible representations contained in the transformation law $D(g)$ of the wave function:

$$\sum_{k=1}^{N}\Lambda_{ik}(s)x_k(s\rho)=\lambda_\alpha x_i(s\rho) \tag{3.20}$$

As already familiar from the atomic and nuclear shell models, the dependence on the magnetic quantum numbers $\sigma, a_i, b_i, \ldots$ can be completely factored out of the dynamical equations; only the magnitude of the spin s and the other angular momenta $A_i, B_i, \ldots$ involved enter in the form of $6j$ symbols or higher recoupling coefficients. We call $\Lambda(s)$ the "reduced mass matrix"; it acts on the "reduced wave function" $x(\alpha)$ with the N components $x_i(\alpha)$ [We remember that $x(\alpha)$ is, of course, independent of σ].

For the matrix element $\Lambda_{ik}(s)$ to be different from zero (i.e., the Racah coefficient in (2.5)], four triplets of angular momenta have to satisfy the triangle inequality: the two triplets $(A_i, A_k, \frac{1}{2})$ and $(B_i, B_k, \frac{1}{2})$ for the representations to be interlocking, and the two triplets (A_i, B_i, s) and (A_k, B_k, s) demanding the spin s to occur in the $SU(2)$ decomposition of both irreducible representations D^i and D^k of $SL(2,C)$. By a similar argument at least two components $x_i(\alpha)$ have to be nonvanishing to obtain a nontrivial eigenvector of the reduced mass matrix. Hence for a particle of spin s and finite mass to occur in the spectrum of a regular wave equation, the spin s has to be contained in the $SU(2)$ decomposition of at least two representations i and k. The matrix β_5 being diagonal, there is a simple relationship between the reduced wave functions $x(\alpha)$ and $y(\alpha)$ belonging to positive and negative eigenvalues $\pm\lambda_\alpha$ of $\Lambda(s)$. Assume

$$\Lambda(s)x(\alpha)=\lambda_\alpha x(\alpha) \tag{3.21a}$$

then

$$\Lambda(s)y(\alpha)=-\lambda_\alpha y(\alpha) \tag{3.21b}$$

with

$$y_i(\alpha)=(-1)^{2A_i}x_i(\alpha), \qquad i=1,2,\ldots,N \tag{3.21c}$$

The matrix elements $\Lambda_{ik}(s)$ are continuous functions of s [see (2.5) and the formulas (C.1) and (C.2) of Appendix C], provided we also define a suitable continuous generalization of the $\Delta(ABs)$ expressing the triangle inequality; we recall that the phases of the $\Lambda_{ik}(s)$ are independent of s. Hence the above eigenvalue equation can trivially be extended to arbitrary real values of the parameter s, and also the eigenvalues $\lambda_\rho(s)$ will be real continuous functions of s. In this way the eigenvalue spectrum of β^0 can naturally be subdivided into various branches which we label by the index ρ. We note that the functions $\lambda_\rho(s)$ will in general neither be strictly positive nor negative definite, i.e., $\lambda_\rho(s)$ may cross over from positive to negative values and vice versa. (For regular wave equations β^0 is nonsingular and therefore $\lambda=0$ can only occur for unphysical values of s.) The masses $m_\alpha=1/\lambda_\alpha=1/|\lambda_\rho(s)|>0$ of the particles described by the wave equation can then also be arranged into corresponding branches, and analogously for the antiparticles; these branches of the mass spectrum will have infinities whenever $\lambda_\rho(s)$ passes through zero. Qualitatively, the mass spectrum of a wave equation containing N irreducible components in its transformation law $D(g)$ consists of $N/2$ positive branches of finite length plus the corresponding antiparticles. (For odd N the wave equation is always nonregular with at least one branch of the mass spectrum at infinity corresponding to vanishing eigenvalues of β^0). The various branches may overlap; some of them may have infinities. The spin dependence within one branch is determined by a certain function of Racah coefficients. [For simple examples of mass spectra see Biritz (1975b).] For $N=2$ the spin dependence of the mass spectrum is rigidly determined by a single Racah coefficient: There is only one free parameter in the theory, the product of the linkage parameters $b_{12}b_{21}$, which may be chosen to arbitrarily fix one mass of the spectrum; all the other mass values are then uniquely determined. We find two qualitatively different types of wave equations with

$N=2$:

Type I: $1=(A,B)$, $2=(A+\frac{1}{2},B+\frac{1}{2})$ leads to a mass spectrum that is increasing with spin, whereas

Type II: $1=(A,B)$, $2=(A+\frac{1}{2},B-\frac{1}{2})$ gives a decreasing mass spectrum.

[These two types of wave equations are found to differ also in other physical properties such as magnetic moments or the renormalization of the axial vector coupling constants (Biritz, 1975c, d).] The aversion against more general wave equations may partly stem from the unfortunate fact that only wave equations with rather unrealistic mass spectra have been considered up to now. For $N>2$ there is a greater flexibility in the mass spectrum. Varying the free parameters b_{ik} corresponding to each linked pair of representations i and k in the β matrices, we can to a large extent influence the shape of the mass spectrum. The b_{ik} are in general overdetermined by the mass spectrum, i.e., there are in general more states in the theory than there are free parameters; hence not every arbitrarily given mass spectrum can be fitted by a given type of wave equation [note the self-consistency condition (2.12) on the matrix β^0].

For every eigenvector $u(\alpha)$ of β^0 there exists a corresponding reduced wave function $x(\alpha)$ which is an eigenvector of the reduced mass matrix $\Lambda(s)$ belonging to the same eigenvalue λ_α. The reverse is not always true. The N-dimensional matrix $\Lambda(s)$ may have up to N nontrivial and linearly independent eigenvectors $x(\alpha),y(\alpha)$; however, not all of them necessarily lead to nontrivial and linearly independent eigenvectors $u(\alpha),v(\alpha)$ of β^0 as some or all of the spin projectors $\chi_i(s)$ in (3.19) may be identically zero. Physically, β^0 need not exactly have N states for every spin value in its spectrum as the various branches of its mass spectrum may start and end at different spin values. This does not cause any problems as we can always choose a suitable basis of reduced wave functions $x(\alpha),y(\alpha)$ such that the supernumerary, i.e., linearly dependent vectors $u(\alpha),v(\alpha)$ vanish identically.

Having solved the momentum space wave equation in the rest frame, we obtain the spinors $u(p\alpha)$ describing particles in motion by applying the boosts $[p\alpha]$. We define

$$u(p\alpha)=N_\alpha D([p\alpha])u(\alpha) \tag{3.22a}$$

and

$$v(p\alpha)=N_\alpha D([p\alpha])v(\alpha)=\beta_5 u(p\alpha) \tag{3.22b}$$

where N_α is a normalization factor to be chosen below. Of course, the boosts $[p\alpha]$ depend on the masses m_α since, for a given momentum, $\mathbf{p}$

particles with different masses move with different velocities. The rapidity ζ_α of the particular boost employed in (3.22a) and (3.22b) is determined by $\sinh\zeta_\alpha = |\mathbf{p}|/m_\alpha$, or $\cosh\zeta_\alpha = E_\alpha/m_\alpha$, with $E_\alpha(\mathbf{p}) = {}_{+}(\mathbf{p}^2 + m_\alpha^2)^{1/2}$. The so-defined spinors satisfy the appropriate wave equation in momentum space: We write the eigenvalue equation (3.17a) as $m_\alpha \beta^0 u(\alpha) = u(\alpha)$, or $\beta_\mu ({}_\alpha p_r)^\mu u(\alpha) = u(\alpha)$, ${}_\alpha p_r = (m_\alpha, 0, 0, 0)$ being the four-momentum of the particle at rest. Applying the boost $D([p\alpha])$ on both sides of the equation and using the transformation properties of the β matrices, also that by definition $L([p\alpha])_\alpha p_r = {}_\alpha p = (E_\alpha, \mathbf{p})$, we obtain the wave equation

$$({}_\alpha \not{p} - \mathbf{1}) u(p\alpha) = 0 \tag{3.23a}$$

with $\not{p} = p_\mu \beta^\mu$. Similarly the spinors $v(p\alpha)$ are found to obey the wave equation

$$({}_\alpha \not{p} + \mathbf{1}) v(p\alpha) = 0 \tag{3.23b}$$

Finally, we define the wave functions in coordinate space

$$f_+(x|p\alpha) = u(p\alpha)\exp(-ip\cdot x) \tag{3.24a}$$

$$f_-(x|p\alpha) = v(p\alpha)\exp(+ip\cdot x) \tag{3.24b}$$

for simplicity we have suppressed the index α in $p = {}_\alpha p$. All these plane wave functions $f_\pm$ are solutions of the wave equation (1.5),

$$(-i\not{\partial} + \mathbf{1}) f_\pm(x|p\alpha) = 0 \tag{3.24c}$$

The corresponding orthogonality and completeness relations will be studied in the next section.

These wave functions have under the Poincaré group the transformation properties (3.5a)–(3.5c) characteristic of particles with mass $\pm m_\alpha$ and spin s. We define the operators $T_{a,g}$ representing (active) Poincaré transformations (a,g) as

$$(T_{a,g}\psi)(x') = D(g)\psi(x) \tag{3.25}$$

where $x' = L(g)x + a$. The plane waves $f_\pm$ are then found to transform under translations as

$$T_a f_\pm(x|p\alpha) = \exp(\pm ip\cdot a) f_\pm(x|p\alpha) \tag{3.26a}$$

for Lorentz transformations we obtain

$$T_g f_\pm(x|ps\sigma\rho) = f_\pm(x|p's\sigma'\rho) D^s[W(g,p\alpha)]_{\sigma'\sigma} \tag{3.26b}$$

with $p'=L(g)p$, the usual Wigner rotation $W(g,p\alpha)=[p'\alpha]^{-1}g[p\alpha]$, and a sum over the double index σ'. The physical significance of the polarization index σ again depends on the particular type of boost employed.

We emphasize that in the last equation the spin s and the degeneracy parameter ρ are not transformed. Repeatedly the objection has been raised that "the traditional method of solving the equation in the rest frame and then boosting to arbitrary momentum may not be applicable to multimass, multispin equations" as, so the saying goes, "the boosts may mix the various spins." This argument seems to be based on an unwarranted analogy with the "nonrelativistic" (i.e., Galilean-relativistic) Pauli spin theory. It is certainly true that a Lorentz transformation $D(g)$ will mix the spin projectors. For example, if we expand the wave function $u_i(p\alpha)$ in terms of the spin projectors $\chi_i(s\sigma)$ referring to the rest frame, in general all the different angular momenta contained in the $SU(2)$ decomposition of D^i will contribute,

$$u_i(ps\sigma\rho)=\sum_{s'} c_i(ps\rho|s')\chi_i(s'\sigma) \tag{3.27}$$

i fixed; only in the limit $\mathbf{p}\rightarrow 0$ do we find the c_i to be proportional to $\delta_{ss'}$. However, this has to be so: If Lorentz transformations would not mix the spin projectors $\chi_i(s)$, we would be able to completely separate the momentum and spin dependence in the wave function, a result that only holds in Galilean relativity but not for truly (Einsteinian) relativistic wave functions, due to the well-known spin–orbit coupling. After all, the spin of a particle is not determined by the expansion of the wave function $u(p\alpha)$ in terms of the spin projectors referring to the rest frame; what actually matters is that, according to the general formulas (3.5), the $u(ps)$ transform among themselves,

$$D(g)u(ps\sigma\rho)=u(p's\sigma'\rho)D^s(W)_{\sigma'\sigma} \tag{3.28}$$

This equation at once follows from the definition of the $u(p\alpha)$, equations (3.22a) and (3.22b) and the fact that the $u(\alpha)$ transform among themselves under rotations, (3.18a). To put it simply, Lorentz transformations necessarily mix the spin projectors $\chi_i(s)$ but they certainly do not mix the spins of the various one-particle wave functions $f_{\pm}(x|p\alpha)$. To obtain these plane wave solutions of the wave equation we can quite literally follow the traditional method of first solving the wave equation in the rest frame and then boosting to arbitrary momentum; we only have to be careful to use different boosts $[p\alpha]$ for states having different masses m_α.

4. ADJOINT WAVE EQUATION AND WAVE FUNCTION; SCALAR PRODUCT, ORTHOGONALITY AND COMPLETENESS RELATIONS

Having standardized the transformation law $D(g)$ of the wave function, we cannot impose any further conditions, like hermiticity of β^0 or the existence of a hermitizing matrix, without severely and unduly restricting the class of wave equations to be considered. From the theoretical point of view there is no need for additional demands on $D(g)$ or the matrices β^μ. In this section we want to show that for any regular wave equation we can in a natural way define the adjoint wave function and a conserved Lorentz-invariant scalar product (positive definite in Fock space), essential for the quantum mechanical formalism.

For every wave equation (1.5) we define the corresponding adjoint wave equation to be

$$\bar{\psi}(x)\left(i\beta^\mu\overleftarrow{\partial}_\mu+\mathbf{1}\right)=0 \tag{4.1}$$

from this rather obvious definition all other developments will be seen to follow inevitably.

As the behavior of the matrices β^μ under Lorentz transformations has already been fixed by (1.3), the row vector $\bar{\psi}(x)$ necessarily has to transform under the Poincaré group according to

$$\bar{\psi}'(x')=\bar{\psi}(x)D^{-1}(g) \tag{4.2}$$

for all $x'=L(g)x+a$. Evidently the adjoint wave equation has exactly the same mass and spin spectrum as the original wave equation. In the basis of the plane wave solutions the connection between a particular wave function $f_\pm(x|p\alpha)$ and its corresponding adjoint wave function $\bar{f}_\pm(x|p\alpha)$ is most easily established in the rest frame. For a regular wave equation (1.5) containing only massive particles β^0 is required to have a complete set of eigenvectors belonging to real eigenvalues; hence β^0 is equivalent to a real diagonal matrix d:

$$M^{-1}\beta^0M=d \tag{4.3}$$

d being real, we also have that

$$(\beta^0)^\dagger=(MM^\dagger)^{-1}\beta^0(MM^\dagger) \tag{4.4}$$

The similarity transformation M will in general not be unitary as β^0 is not required to be Hermitian. [The fact that β^0 is equivalent to its Hermitian conjugate does not generally imply the existence of a hermitizing matrix η with $(\beta^\mu)^\dagger = \eta\beta^\mu\eta^{-1}$; for that also $D(g^{-1})^\dagger = \eta D(g)\eta^{-1}$ would be necessary, but here we are not making such an assumption about the transformation law $D(g)$.] We denote by $t(\alpha\epsilon)$ the eigenvectors of d,

$$dt(\alpha\epsilon) = \epsilon\lambda_\alpha t(\alpha\epsilon) \tag{4.5a}$$

where $\epsilon = \pm 1$ distinguishes the solutions belonging to positive and negative eigenvalues. They satisfy the orthogonality relations

$$t^\dagger(\alpha\epsilon)t(\alpha'\epsilon') = \delta(\alpha,\alpha')\delta_{\epsilon\epsilon'} \tag{4.5b}$$

and the completeness relation

$$\sum_{\alpha\epsilon} t(\alpha\epsilon)\otimes t^\dagger(\alpha\epsilon) = \mathbf{1} \tag{4.5c}$$

Naturally we have used in (4.5b) the usual unitary and positive definite scalar product for the eigenvectors $t(\alpha\epsilon)$; it would be quite irksome to do otherwise. Applying the similarity transformation M we obtain for the (right) eigenvectors $w(\alpha\epsilon)$ of β^0

$$w(\alpha\epsilon) = Mt(\alpha\epsilon) \tag{4.6a}$$

$$\beta^0 w(\alpha\epsilon) = \epsilon\lambda_\alpha w(\alpha\epsilon) \tag{4.6b}$$

where $w(\alpha\epsilon)$, $\epsilon = \pm 1$, collectively stands for the spinors $u(\alpha)$ and $v(\alpha)$ discussed in Section 3. For every $w(\alpha\epsilon)$ we define its corresponding adjoint (left) eigenvector $\bar{w}(\alpha\epsilon)$ of β^0 as

$$\bar{w}(\alpha\epsilon) = t^\dagger(\alpha\epsilon)M^{-1} \tag{4.6c}$$

$$\bar{w}(\alpha\epsilon)\beta^0 = \epsilon\lambda_\alpha \bar{w}(\alpha\epsilon) \tag{4.6d}$$

For the definition of the adjoint spinor $\bar{w}(\alpha\epsilon)$ we have no choice but the matrix M^{-1} in (4.6c), as in general $w^\dagger(\alpha\epsilon) = t^\dagger(\alpha\epsilon)\ M^\dagger$ will not be an eigenvector of β^0. The following orthogonality and completeness relations hold:

$$\bar{w}(\alpha\epsilon)w(\alpha'\epsilon') = \delta(\alpha,\alpha')\delta_{\epsilon\epsilon'} \tag{4.7a}$$

$$\sum_{\alpha\epsilon} w(\alpha\epsilon)\otimes\bar{w}(\alpha\epsilon) = \mathbf{1} \tag{4.7b}$$

compared to

$$\sum_{\alpha\epsilon} w(\alpha\epsilon)\otimes w^{\dagger}(\alpha\epsilon)=MM^{\dagger} \tag{4.7c}$$

In Section 3 we derived that

$$\beta_5 w(\alpha\epsilon)=w(\alpha,-\epsilon) \tag{4.8a}$$

similarly we have for the adjoint wave functions

$$\bar{w}(\alpha\epsilon)\beta_5=\bar{w}(\alpha,-\epsilon) \tag{4.8b}$$

as the matrix β_5 commutes with $MM^{\dagger}$ [see Eq. (4.7c)]. Analogously $MM^{\dagger}$ commutes with $D(r)$ for all rotations $r\in SU(2)$, and we find that the adjoint wave functions $\bar{w}(\alpha\epsilon)$ transform under rotations similarly to the $w(\alpha\epsilon)$:

$$D(r)w(s\sigma\rho,\epsilon)=w(s\sigma'\rho,\epsilon)D^{s}(r)_{\sigma'\sigma} \tag{4.9a}$$

$$\bar{w}(s\sigma\rho,\epsilon)D(r)=D^{s}(r)_{\sigma\sigma'}\bar{w}(s\sigma'\rho,\epsilon) \tag{4.9b}$$

with a sum over the double index σ'. Expanding the spinors w ($\bar{w}$) in terms of the spin projectors $\chi_i(s)$ introduced in (2.2), we obtain the N constants z_i ($\bar{z}_i$) which make up the components of the reduced wave function:

$$w_i(s\sigma\rho,\epsilon)=z_i(s\rho\epsilon)\chi_i(s\sigma) \tag{4.10a}$$

$$\bar{w}_i(s\sigma\rho,\epsilon)=\bar{z}_i(s\rho\epsilon)\chi_i^{\dagger}(s\sigma) \tag{4.10b}$$

where we have suppressed the indices a_ib_i. The above equations (4.6) and (4.7) are then easily translated into relations between the reduced wave functions and the reduced mass matrix $\Lambda(s)$ defined in (2.5):

$$\sum_k \Lambda_{ik}(s)z_k(s\rho\epsilon)=\epsilon\lambda_\alpha z_i(s\rho\epsilon) \tag{4.11a}$$

$$\sum_i \bar{z}_i(s\rho\epsilon)\Lambda_{ik}(s)=\epsilon\lambda_\alpha \bar{z}_k(s\rho\epsilon) \tag{4.11b}$$

$i,k=1,2,\ldots,N$. For fixed s we have the orthogonality relations

$$\sum_i \bar{z}_i(s\rho\epsilon)z_i(s\rho'\epsilon')\Delta(A_iB_is)=\delta_{\rho\rho'}\delta_{\epsilon\epsilon'} \tag{4.12}$$

and, again for every s, the completeness relation

$$\sum_{\rho\epsilon} z_i(s\rho\epsilon)\bar{z}_k(s\rho\epsilon)\Delta(A_iB_is)\Delta(A_kB_ks)=\delta_{ik} \tag{4.13}$$

We observe that in these orthogonality and completeness relations actually only those reduced wave functions contribute that give rise to nonvanishing wave functions $w(\alpha\epsilon)$ or $\bar{w}(\alpha\epsilon)$, i.e., where the appropriate triangle inequalities are satisfied: $\Delta(j_1j_2j_3)=1$ if the triplet of angular momenta satisfies the triangle inequality, otherwise $\Delta=0$. From (4.8) we derive the simple relation between the reduced wave functions belonging to opposite eigenvalues of $\Lambda(s)$,

$$z_i(\alpha,-\epsilon)=(-1)^{2A_i}z_i(\alpha,\epsilon) \tag{4.14a}$$

and also

$$\bar{z}_i(\alpha,-\epsilon)=(-1)^{2A_i}\bar{z}_i(\alpha,\epsilon) \tag{4.14b}$$

The knowledge of the reduced wave functions is essential not only for the computation of the mass spectrum but also for all practical calculations as the $z_i(\alpha\epsilon)$ and $\bar{z}_k(\alpha'\epsilon')$ explicitly occur in the formulas of physical observables [As an example see Biritz (1975c) for the expression of the magnetic moments.] These reduced wave functions $z(\alpha\epsilon)$ together with their corresponding adjoints $\bar{z}(\alpha\epsilon)$ are determined as the right and left eigenvectors of the reduced mass matrix $\Lambda(s)$, that is by a simple N-dimensional eigenvalue problem; we recall that N is the number of irreducible representations $D^i(g)$ of $SL(2,C)$ contained in the transformation law $D(g)$.

The spinors $w(p\alpha\epsilon)$ describing particles in motion are obtained by applying the boost $[p\alpha]$ appropriate for the state α,

$$w(p\alpha\epsilon)=N_\alpha D([p\alpha])w(\alpha\epsilon) \tag{4.15a}$$

The adjoint wave function transforms under the Lorentz group according to $D^{-1}(g)$, equation (4.2); hence the adjoint spinor corresponding to $w(p\alpha\epsilon)$ is given by

$$\bar{w}(p\alpha\epsilon)=N_\alpha\bar{w}(\alpha\epsilon)D^{-1}([p\alpha]) \tag{4.15b}$$

In these formulas N_α is a normalization constant to be chosen presently. We observe that the adjoint wave function $\bar{w}(p\alpha\epsilon)$ will in general not be simply related to the Hermitian conjugate of $w(p\alpha\epsilon)$. Such a relation only holds in the rest frame where $\bar{w}(\alpha\epsilon)=w^\dagger(\alpha\epsilon)(MM^\dagger)^{-1}$; however, as we do not impose any restrictions on $D(g)$, this simple connection is generally lost for $\mathbf{p}\neq 0$.

Before discussing the adjoint wave functions in coordinate space, we first want to derive the orthogonality and completeness relations in momentum space. From the wave equations satisfied by $w(p\alpha\epsilon)$ and its adjoint,

$$(\epsilon_\alpha \not{p} - \mathbf{1})w(p\alpha\epsilon)=0 \tag{4.16a}$$

$$\bar{w}(p\alpha\epsilon)(\epsilon_\alpha \not{p} - \mathbf{1})=0 \tag{4.16b}$$

we at once deduce for the matrix elements

$$\bar{w}(\epsilon\mathbf{p},\alpha\epsilon)\beta^0 w(\epsilon'\mathbf{p},\alpha'\epsilon')\sim\delta_{m_\alpha m_{\alpha'}}\delta_{\epsilon\epsilon'} \tag{4.17a}$$

There $\epsilon\mathbf{p}$ stands for the 3-momentum $\pm\mathbf{p}$; we reserve the letters w and $\bar{w}$ for the wave functions on the mass shell and can then omit the corresponding energy $E_\alpha(\mathbf{p})$. As the boost $[p\alpha]$ only depends on the mass m_α (and not, say, on the spin s or the degeneracy parameter ρ), the nonvanishing matrix elements in (4.17a) are related to matrix elements of the β^μ between the spinors at rest by a simple Lorentz transformation:

$$\bar{w}(\epsilon\mathbf{p},\alpha\epsilon)\beta^0 w(\epsilon'\mathbf{p},\alpha'\epsilon')=\delta_{m_\alpha m_{\alpha'}}\delta_{\epsilon\epsilon'}N_\alpha^2 L([p\alpha])^0_{\ \mu}\bar{w}(\alpha\epsilon)\beta^\mu w(\alpha'\epsilon') \tag{4.17b}$$

From the commutation relations (2.11b) of the $\boldsymbol{\beta}$ with the generators of the Lorentz group we infer that

$$\bar{w}(\alpha\epsilon)\boldsymbol{\beta}w(\alpha'\epsilon')=i\epsilon(\lambda_{\alpha'}-\lambda_\alpha)\bar{w}(\alpha\epsilon)\mathbf{K}w(\alpha'\epsilon') \tag{4.17c}$$

with $\lambda_\alpha=1/m_\alpha$. Hence only $\mu=0$ contributes to the matrix elements in (4.17b), and we arrive at the familiar orthogonality relations, written here separately for the spinors $u(\mathbf{p}\alpha)$ and $v(\mathbf{p}\alpha)$ belonging to positive and negative frequencies:

$$\bar{u}(\mathbf{p}\alpha)\beta^0 u(\mathbf{p}\alpha')=2E_\alpha(\mathbf{p})\delta(\alpha,\alpha') \tag{4.18a}$$

$$\bar{v}(\mathbf{p}\alpha)\beta^0 v(\mathbf{p}\alpha')=-2E_\alpha(\mathbf{p})\delta(\alpha,\alpha') \tag{4.18b}$$

$$\bar{u}(\mathbf{p}\alpha)\beta^0 v(-\mathbf{p}\alpha')=\bar{v}(\mathbf{p}\alpha)\beta^0 u(-\mathbf{p}\alpha')=0 \tag{4.18c}$$

where we have chosen for the normalization constant

$$N_\alpha^2=2m_\alpha^2 \tag{4.18d}$$

For fixed $\mathbf{p}$, all the $u(\mathbf{p}\alpha)$ together with the $v(-\mathbf{p}\alpha)$ form a complete set,

$$\sum_{\alpha}[2E_{\alpha}(\mathbf{p})]^{-1}[\bar{u}(\mathbf{p}\alpha)u(\mathbf{p}\alpha)-\bar{v}(-\mathbf{p}\alpha)v(-\mathbf{p}\alpha)]=\beta_0^{-1} \tag{4.19}$$

We define the adjoint functions $\bar{f}(x|p\alpha\epsilon)$ corresponding to the plane wave functions $f(x|p\alpha\epsilon)$ introduced in (3.24) as

$$\bar{f}(x|p\alpha\epsilon)=\bar{w}(p\alpha\epsilon)\exp(i\epsilon_{\alpha}p\cdot x) \tag{4.20}$$

The so-determined adjoint wave functions form a complete set of solutions of the adjoint wave equation

$$\bar{f}(x|p\alpha\epsilon)(i\overleftarrow{\partial\!\!\!/}+\mathbf{1})=0 \tag{4.21}$$

In general, to obtain the adjoint wave function $\bar{\psi}(x)$ for an arbitrarily given solution $\psi(x)$ of the wave equation, we expand $\psi(x)$ in terms of plane waves,

$$\psi(x)=\sum_{\alpha}\int(dp\alpha)[a(p\alpha)f_{+}(x|p\alpha)+b(p\alpha)f_{-}(x|p\alpha)] \tag{4.22a}$$

with $(dp\alpha)=d^3p/2E_{\alpha}(\mathbf{p})$. The adjoint wave function $\bar{\psi}(x)$ corresponding to (4.22a) is now defined as

$$\bar{\psi}(x)=\sum_{\alpha}\int d(p\alpha)\left[a^*(p\alpha)\bar{f}_{+}(x|p\alpha)+b^*(p\alpha)\bar{f}_{-}(x|p\alpha)\right] \tag{4.22b}$$

the asterisk denoting the complex conjugate. Finally we introduce the scalar product for any two solutions ψ and Φ of the wave equation

$$(\psi,\Phi)=\int d^3x\bar{\psi}(x)\beta^0\Phi(x) \tag{4.23a}$$

or more generally

$$(\psi,\Phi)=\int d\sigma_{\mu}\bar{\psi}(x)\beta^{\mu}\Phi(x) \tag{4.23b}$$

with the integration extending over an arbitrary spacelike surface. In particular we obtain for the plane wave solutions

$$(f(p\alpha\epsilon),f(p'\alpha'\epsilon'))=\epsilon(2\pi)^3 2E_{\alpha}(\mathbf{p})\delta(\mathbf{p}-\mathbf{p}')\delta(\alpha,\alpha')\delta_{\epsilon\epsilon'} \tag{4.24}$$

and thus in the general case

$$(\psi,\Phi)=\sum_{\alpha}\int(dp\alpha)\{a^*(p\alpha)c(p\alpha)-b^*(p\alpha)d(p\alpha)\} \tag{4.25}$$

where, analogously to (4.22a), the c and d are the expansion coefficients of $\Phi(x)$ in terms of the plane waves.

This scalar product fulfills all the usual requirements:

(i) It is linear in the second factor,

$$(\psi,\gamma_1\Phi_1+\gamma_2\Phi_2)=\gamma_1(\psi,\Phi_1)+\gamma_2(\psi,\Phi_2) \tag{4.26}$$

(ii) It is Hermitian,

$$(\psi,\Phi)=(\Phi,\psi)^* \tag{4.27}$$

(iii) $\Phi(x)$ and $\bar{\psi}(x)$ being any solutions of the wave equation and its adjoint it follows that

$$\partial_\mu\left[\bar{\psi}(x)\beta^\mu\Phi(x)\right]=0 \tag{4.28}$$

Hence the scalar product does not depend on the special choice of the space-like surface used in (4.23b), and in particular the scalar product is time independent, as already evident from the explicit expression given in (4.25).

(iv) This last result together with the behavior of the wave functions and the β^μ under the Poincaré group implies that the scalar product is Lorentz invariant, i.e., that the transformations $T_{a,g}$ defined in (3.25) are unitary within this scalar product:

$$(T_{a,g}\psi,T_{a,g}\Phi)=(\psi,\Phi) \tag{4.29}$$

It is apparent from (4.24) or (4.25) that the scalar product is indefinite. Furthermore the Hamiltonian

$$H=\beta_0^{-1}\{-i\boldsymbol{\beta}\cdot\boldsymbol{\nabla}+1\} \tag{4.30}$$

although Hermitian in this scalar product, is not positive definite,

$$Hf(x|p\alpha\epsilon)=\epsilon E_\alpha(\mathbf{p})f(x|p\alpha\epsilon) \tag{4.31}$$

and therefore without a lower bound. These are already familiar difficulties encountered in c-number theories, and it is well known that a consistent physical interpretation of relativistic wave equations is only possible within the framework of a quantum field theory. We shall see in Section 8 that the second-quantized theories based on such wave equations have a Hermitian and positive definite Hamilton operator, as well as a positive definite metric in Fock space.

Here our main concern was the existence of a uniquely defined adjoint wave function $\bar{\psi}(x)$ with the usual properties, even for theories which do not permit a hermitizing matrix, that is where the transformation law $D(g)$ is not pseudounitary. The establishment of the quantum mechanical formalism does not depend on a hermitizing matrix (in this context see also Weaver, Hammer, and Good, 1964; Hurley, 1974); more importantly we prefer our definition of the adjoint wave function even in those cases where there exists a hermitizing matrix, as in this way we can avoid an indefinite metric in Fock space (see Sections 7 and 8). Our definition of the adjoint wave function appears to be somewhat cumbersome for the c-number theory: We first have to expand ψ in terms of plane waves, and then find $\bar{\psi}$ according to (4.22b). However, this is just what is needed in the corresponding quantum field theory where the expansion coefficients are interpreted as creation and absorption operators. The so-defined adjoint field operators are not more difficult to use than those based on the usual hermitizing matrix: For the evaluation of the mass spectrum the reduced wave functions $z(\alpha\epsilon)$ have to be computed in any case, being given as the (right) eigenvectors of the reduced mass matrix $\Lambda(s)$. It is then only a minor complication that the adjoint reduced wave functions $\bar{z}(\alpha\epsilon)$ are not simply related to the complex-conjugate transpose of $z(\alpha\epsilon)$ but have to be separately determined as the left eigenvectors of $\Lambda(s)$, subject to the normalization condition (4.12). More generally the connection between $\psi(x)$ and its adjoint wave function $\bar{\psi}(x)$ can be written in the form

$$\bar{\psi}(x)=\left[\eta(i\partial)\psi(x)\right]^{\dagger} \tag{4.32}$$

where $\eta(i\partial)$ is a certain differential operator the explicit form of which shall not concern us here. One might argue that there is no "local" connection between ψ and $\bar{\psi}$. There is no need for one. What actually matters is that the so-defined adjoint field operator $\bar{\psi}(x)$ has local (anti-)commutation relations with $\psi(x)$; this also will be deferred to Section 8.

5. THE KLEIN–GORDON DIVISOR AND PROJECTION OPERATORS

The various plane wave solutions $f_{\pm}(x|p\alpha)$ describe particles with definite masses m_α and as such trivially satisfy the corresponding Klein–Gordon equations. Clearly there exists an intimate connection between the solutions of the wave equation (1.5) and those of an appropriate set of Klein–Gordon equations. Formally this is expressed in terms of the Klein–Gordon divisor $d(q)$, a uniquely determined matrix polynomial in the 4-vector q^μ, satisfying the algebraic identity

$$(-\not{q}+\mathbf{1})d(q)=d(q)(-\not{q}+\mathbf{1})=\prod_{k=1}^{n}(-q^2+m_k^2)\mathbf{1} \qquad (5.1)$$

There $\not{q}=q_\mu\beta^\mu$, $q^2=q_\mu q^\mu$, with q^μ being an arbitrary complex 4-vector; we reserve the letter q for general 4-momenta off the mass shell. The product on the right-hand side of the equation goes over all the different masses m_k contained in the spectrum of the wave equation (1.5) irrespectively of their multiplicities, i.e., independently of spin or further degeneracies. As some of the states $\alpha=(s\rho)$ may have equal mass, we use the index $k=1,2,\ldots,n$ for labeling the n distinct masses m_k of the wave equation, to indicate clearly that only the different masses contribute to the Klein–Gordon divisor. In this section we want to derive three formulas for $d(q)$, as the knowledge of the Klein–Gordon divisor is of considerable practical and theoretical importance: For example, by means of $d(q)$ we will be able to express the various invariant functions of the wave equation (1.5) in terms of the corresponding functions associated with a set of Klein–Gordon equations.

Evidently the Klein–Gordon divisor is related to the inverse of the matrix $\mathbf{1}-\not{q}$; for this purpose we briefly review some simple facts about matrix algebra (Smirnov, 1964). Any diagonalizable matrix X (in our case β^0) satisfies the minimal equation

$$(X-\xi_1)(X-\xi_2)\cdots(X-\xi_n)=0 \qquad (5.2)$$

ξ_k being all the different eigenvalues of X, $\xi_i\neq\xi_k$ for $i\neq k$; this relation is easily proved by applying a similarity transformation which brings X into diagonal form. We define the minimal polynomial

$$\mu(z)=(z-\xi_1)(z-\xi_2)\cdots(z-\xi_n) \qquad (5.3)$$

and the related polynomials $\mu_k(z)$, $k=1,2,\ldots,n$, given by

$$\mu(z)=(z-\xi_k)\mu_k(z) \qquad (5.4)$$

Furthermore we introduce the normalized polynomials

$$\mathfrak{S}_k(z)=\mu_k(z)/\mu_k(\xi_k) \qquad (5.5a)$$

which satisfy

$$\mathfrak{S}_i(\xi_k)=\delta_{ik} \qquad (5.5b)$$

These $\mathfrak{S}_k(z)$ form a basis in the space of all polynomials of degree $n-1$. For a given integer g, $0 \leqslant g \leqslant n-1$, consider the expression

$$Q(z) = z^g - \sum_{k=1}^{n} \xi_k \mathfrak{S}_k(z) \tag{5.6}$$

the degree of which is not larger than $n-1$. Actually, as we know n distinct values ξ_i with $Q(\xi_i)=0$, $i=1,2,\ldots,n$, we conclude that $Q(z)$ vanishes identically. Thus

$$z^g = \sum_{k=1}^{n} \xi_k^g \mathfrak{S}_k(z) \tag{5.7}$$

for $g=0,1,\ldots,n-1$. (In the next section we shall deduce from this simple algebraic identity that the contact terms in the propagator vanish.)

By means of these polynomials we can construct a complete set of projection operators $\mathfrak{S}_k(X)$. Equation (5.7) is an algebraic identity which also holds for the matrix X as argument. Taking $g=0$ we find

$$\sum_{k=1}^{n} \mathfrak{S}_k(X) = \mathbf{1} \tag{5.8}$$

$\mu(z)$ being the minimal polynomial of the matrix X implies that $(X-\xi_k)\mu_k(X)=\mu(X)=0$. It follows that

$$X\mathfrak{S}_k(X) = \xi_k \mathfrak{S}_k(X) \tag{5.9}$$

and hence for any function $f(x)$ of the matrix X

$$f(X)\mathfrak{S}_k(X) = f(\xi_k)\mathfrak{S}_k(X) \tag{5.10}$$

In particular we obtain that the $\mathfrak{S}_k(X)$ are projection operators,

$$\mathfrak{S}_i(X)\mathfrak{S}_k(X) = \mathfrak{S}_i(\xi_k)\mathfrak{S}_k(X) = \delta_{ik}\mathfrak{S}_k(X) \tag{5.11}$$

As the $\mathfrak{S}_k(X)$ form a complete set, any function $f(X)$ can be expanded in terms of the $\mathfrak{S}_k(X)$:

$$f(X) = \sum_{k=1}^{n} f(\xi_k)\mathfrak{S}_k(X) \tag{5.12}$$

(sometimes called the formula of Sylvester). For the matrix function $f(X)$ to exist the ordinary complex function $f(z)$ has necessarily to be well defined at the eigenvalues ξ_k of the matrix X; it can be shown (Smirnov,

1964) that this is also the criterion for a power series in the matrix X to converge.

We remark that these projection operators $\mathfrak{S}_k(X)$ are uniquely determined by the following properties: Assume a set of operators (matrices) $\mathfrak{F}_{k,r}$ to be given by

$$X\mathfrak{F}_{k,r}=\xi_k\mathfrak{F}_{k,r} \tag{5.13a}$$

and which satisfy the completeness relation

$$\sum_{k,r}\mathfrak{F}_{k,r}=\mathbf{1} \tag{5.13b}$$

As the $\mathfrak{S}_k(X)$ are simple polynomials in X we deduce that

$$\mathfrak{S}_k(X)\mathfrak{F}_{i,r}=\mathfrak{S}_k(\xi_i)\mathfrak{F}_{i,r}=\delta_{ik}\mathfrak{F}_{i,r} \tag{5.14}$$

which together with the completeness relation (5.13b) implies

$$\mathfrak{S}_k(X)=\sum_r\mathfrak{F}_{k,r} \tag{5.15}$$

For the Klein–Gordon divisor we need the inverse of the matrix $\mathbf{1}-X$. From (5.12) we obtain

$$(\mathbf{1}-X)^{-1}=\sum_{k=1}^{n}(1-\xi_k)^{-1}\mathfrak{S}_k(X) \tag{5.16}$$

where with no loss of generality we may assume all $\xi_k\neq 1$. According to (5.4) we can write $(1-\xi_k)^{-1}=\mu_k(1)/\mu(1)$; in view of the Klein–Gordon divisor we define a matrix polynomial $\mathfrak{D}(X)$,

$$\mathfrak{D}(X)=\sum_{k=1}^{n}\mu_k(1)\mathfrak{S}_k(X) \tag{5.17}$$

which satisfies

$$(\mathbf{1}-X)\mathfrak{D}(X)=\mu(1)\mathbf{1} \tag{5.18}$$

where $\mu(1)=\Pi(1-\xi_k)$ from (5.3). Another useful expression for the Klein–Gordon divisor is obtained by explicitly writing $\mathfrak{D}(X)$ as a polynomial in X. The corresponding expansion coefficients are given in terms of the elementary symmetric functions e_k built from the n roots of the minimal polynomial $\mu(z)$. They are defined as follows (Perron, 1951):

$$\begin{aligned}\mu(z)&=(z-\xi_1)(z-\xi_2)\cdots(z-\xi_n)\\&=z^n+e_1z^{n-1}+\cdots+e_kz^{n-k}+\cdots+e_n\end{aligned}\tag{5.19}$$

We find

$$\begin{aligned}e_1&=-(\xi_1+\xi_2+\cdots+\xi_n)\\e_2&=\xi_1\xi_2+\xi_1\xi_3+\cdots+\xi_{n-1}\xi_n\\&\vdots\\e_n&=(-1)^n\xi_1\xi_2\cdots\xi_n\end{aligned}\tag{5.20a}$$

with the general term given by

$$e_k=(-1)^k\sum\xi_{i_1}\xi_{i_2}\cdots\xi_{i_k}\tag{5.20b}$$

the sum going over all combinations of indices which satisfy $i_1<i_2<\cdots<i_k$. It is also useful to define $e_0=1$. We obtain for $\mathfrak{D}(X)$ the expansion

$$\mathfrak{D}(X)=\sum_{k=1}^{n}d_kX^{n-k}$$

with the coefficients

$$d_k=e_0+e_1+\cdots+e_{k-1}\tag{5.21b}$$

This relation either follows from rearranging (5.17) or, more simply, by directly proving that the matrix $\mathfrak{D}(X)$ as defined in (5.21) actually satisfies $(\mathbf{1}-X)\mathfrak{D}(X)=\mu(1)\mathbf{1}$.

The matrix β^0 is diagonalizable for a regular wave equation (1.5) describing only massive particles. Therefore β^0 has the minimal equation

$$\prod_{k=1}^{n}(\beta_0^2-\lambda_k^2)=0\tag{5.22}$$

where the product only contains the different eigenvalues $\lambda_k=1/m_k$ of β^0 irrespectively of their multiplicities; we have used the fact that with every λ_α also $-\lambda_\alpha$ is an eigenvalue. From the behavior of the β^μ under Lorentz transformations we deduce that the matrix $X=\not{q}^2$ ($\not{q}=q_\mu\beta^\mu$, q_μ being an arbitrary complex 4-vector) satisfies the minimal equation $\prod(X-\xi_k)=0$ with $\xi_k=q^2\lambda_k^2$, $q^2=q_\mu q^\mu$. In this case we find

$$\prod(-q^2+m_k^2)=c\mu(1) \tag{5.23a}$$

where we have introduced the constant

$$c=\prod_{k=1}^{n} m_k^2 \tag{5.23b}$$

Here again the product only goes over all the different masses m_k described by the wave equation. By definition the Klein–Gordon divisor satisfies

$$(-\not{q}+\mathbf{1})d(q)=c\mu(1)\mathbf{1} \tag{5.24a}$$

hence

$$\begin{aligned} d(q)&=c\mu(1)(\mathbf{1}-\not{q})^{-1}\\ &=c(\mathbf{1}+\not{q})\mu(1)(\mathbf{1}-\not{q}^2)^{-1} \end{aligned} \tag{5.24b}$$

Setting $X=\not{q}^2$ in (5.18) we immediately derive

$$d(q)=c(\mathbf{1}+\not{q})\mathfrak{D}(\not{q}^2) \tag{5.25}$$

From the above formulas we obtain two different expressions for the Klein–Gordon divisor. We learn from (5.17) that

$$d(q)=(1+\not{q})\sum_{k=1}^{n} c\mu_k(1)\mathcal{E}_k(\not{q}^2) \tag{5.26a}$$

with

$$c\mu_k(1)=m_k^2\prod_{j\neq k}(-q^2+m_j^2) \tag{5.26b}$$

and

$$\mathcal{E}_k(\not{q}^2)=\prod_{j\neq k}\frac{\left[\not{q}/(q^2)^{1/2}\right]^2-\lambda_j^2}{\lambda_k^2-\lambda_j^2} \tag{5.26c}$$

$\lambda_k = 1/m_k$. By means of (5.21) we arrive at an expansion of the Klein–Gordon divisor in powers of $\not{q}$:

$$d(q) = (\mathbf{1} + \not{q}) \sum_{k=1}^{n} cd_k(\not{q}^2)^{n-k} \tag{5.27a}$$

with the coefficients d_k given in (5.21b), where the elementary symmetric functions e_j are to be constructed from the values $\xi_k = q^2/m_k^2$. We can factorize their q^2 dependence and find, with $c = \Pi m_k^2$,

$$ce_k = (-1)^n q^{2k} \tilde{e}_{n-k} \tag{5.27b}$$

$\tilde{e}_j$ denoting the symmetric functions directly built from the n distinct masses m_k^2. As an illustration we quote the simple examples $n = 1, 2, 3$ from which the general building principle of the Klein–Gordon divisor will be obvious:

$n=1$: $$d(q) = (\mathbf{1} + \not{q}) m^2 \tag{5.28a}$$

$n=2$: $$d(q) = (\mathbf{1} + \not{q})\left[m_1^2 m_2^2 \not{q}^2 + m_1^2 m_2^2 - (m_1^2 + m_2^2) q^2 \right] \tag{5.28b}$$

$n=3$: $$d(q) = (\mathbf{1} + \not{q})\left[-\tilde{e}_3(\not{q}^2)^2 - (\tilde{e}_3 + q^2\tilde{e}_2)\not{q}^2 - (\tilde{e}_3 + q^2\tilde{e}_2 + q^4\tilde{e}_1) \right] \tag{5.28c}$$

with $\tilde{e}_1 = -(m_1^2 + m_2^2 + m_3^2)$, $\tilde{e}_2 = m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2$ and $\tilde{e}_3 = -m_1^2 m_2^2 m_3^2$.

The two formulas (5.26) and (5.27) for the Klein–Gordon divisor are not always very useful. For example, in the perturbation expansion of the S matrix one encounters matrix elements of $d(q)$ between certain spinors $w(p\alpha\epsilon)$ describing asymptotically free particles. For this purpose an expression for $d(q)$ in terms of projection operators constructed from the wave functions $w(p\alpha\epsilon)$ is more practical. Corresponding to every state $\alpha = (s\rho)$ in the spectrum of the wave equation we define the operator $\Gamma_+^\alpha(q)$ by its graphical expression given in Figure 6. Suppressing all indices $a_i b_i$, its (ik) block of matrix elements reads

$$\Gamma_+^\alpha(q)_{ik} = \sum_\sigma D^{A_i}[(q)] u_i(\alpha) \otimes \bar{u}_k(\alpha) D^{A_k}[(q)^{-1}] \tag{5.29}$$

Consistent with our notation introduced in Section 2 for the generators of the Lorentz group, we have simply written D^{A_i} for the matrix $D^{A_i} \otimes \mathbf{1}^{B_i}$ and similarly for D^{A_k}, in accordance with Figure 6. In this formula $u(\alpha)$ denotes the eigenvector of β^0 and $\bar{u}(\alpha)$ its adjoint spinor, as derived in Section 3. With an arbitrary complex 4-vector q^μ ($q^2 \neq 0$) we associate the

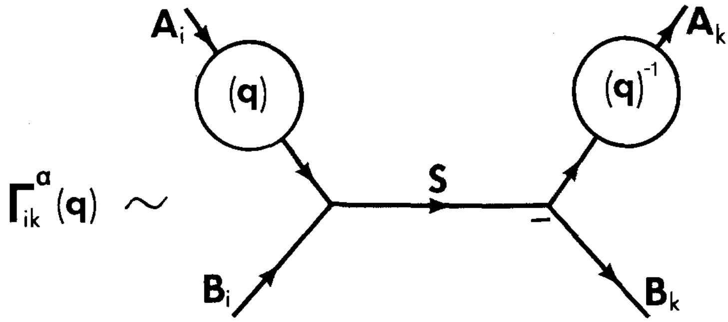


Fig. 6. Graphical representation of the projection operator defined in equation (5.29).

$SL(2,C)$ matrix

$$(q)=q_\mu\sigma^\mu/(q^2)^{1/2} \tag{5.30a}$$

and its inverse

$$(q)^{-1}=q_\mu\tilde{\sigma}^\mu/(q^2)^{1/2} \tag{5.30b}$$

with the usual Pauli matrices introduced in Sections 2 and 3. For definiteness the principal value of the square root has to be taken, $0\leqslant \arg(q^2)^{1/2}<\pi$; we shall see that actually the Klein–Gordon divisor is independent of this choice of the sign.

Similarly we define the operators $\Gamma^\alpha_-(q)$ constructed from the eigenvectors $v(\alpha)$:

$$\Gamma^\alpha_-(q)_{ik}=\sum_\sigma D^{A_i}[(q)]v_i(\alpha)\otimes\bar{v}_k(\alpha)D^{A_k}[(q)^{-1}] \tag{5.31}$$

Using the simple connection (3.17b) between the spinors $u(\alpha)$ and $v(\alpha)$, $v_i(\alpha)=(-1)^{2A_i}u_i(\alpha)$, we obtain

$$\Gamma^\alpha_-(q)=\Gamma^\alpha_+(-q) \tag{5.32}$$

In the following we shall simply write $\Gamma^\alpha(q)$ for $\Gamma^\alpha_+(q)$.

On the mass shell of the state α, $q={}_\alpha p$, ${}_\alpha p^2=m_\alpha^2$, ${}_\alpha p^0>0$, the matrix $({}_\alpha p)$ can be factorized in terms of the boosts $[p\alpha]$ introduced in Section 3,

$$({}_\alpha p)=[p\alpha][p\alpha]^\dagger \tag{5.33}$$

These $SL(2,C)$ matrices can now be shifted along the lines of Figure 6 according to the rules laid down in Appendix A. We find that on the mass shell

$$\Gamma^{\alpha}({}_{\alpha}p) = N_{\alpha}^{-2} u(\mathbf{p}\alpha) \otimes \bar{u}(\mathbf{p}\alpha) \tag{5.34}$$

with the normalization constant $N_{\alpha}^2 = 2m_{\alpha}^2$ from (4.18d). As the projection operators $\mathcal{E}_k(\not{q}^2)$ that make up the Klein–Gordon divisor are uniquely determined, we obtain an unambiguous prescription for the continuation of the wave functions $u(p\alpha)\bar{u}(p\alpha)$ in (5.34) off the mass shell. We want to show that our definition given in equations (5.29) and (5.30) is the correct one. It trivially follows from the orthogonality and completeness relations (4.7a) and (4.7b) of the spinors $u(\alpha)$ and $v(\alpha)$ that the $\Gamma^{\alpha}(q)$ form a complete set of projection operators:

$$\Gamma^{\alpha}(q)\Gamma^{\alpha'}(q) = \delta(\alpha,\alpha')\Gamma^{\alpha}(q) \tag{5.35a}$$

$$\Gamma^{\alpha}(q)\Gamma^{\alpha'}(-q) = 0 \tag{5.35b}$$

$$\sum_{\alpha} \Gamma^{\alpha}(q) + \Gamma^{\alpha}(-q) = \mathbf{1} \tag{5.35c}$$

Their importance for the Klein–Gordon divisor stems from the relation

$$\not{q}\Gamma^{\alpha}(q) = \Gamma^{\alpha}(q)\not{q} = (q^2)^{1/2}\lambda_{\alpha}\Gamma^{\alpha}(q) \tag{5.36a}$$

and thus also

$$\not{q}\Gamma^{\alpha}(-q) = -(q^2)^{1/2}\lambda_{\alpha}\Gamma^{\alpha}(-q) \tag{5.36b}$$

This equation is easily proved with the help of the graphical expressions for the matrices β^{μ} and the projection operator Γ^{α}. In particular we find that

$$\not{q}^2[\Gamma^{\alpha}(q) + \Gamma^{\alpha}(-q)] = q^2\lambda_{\alpha}^2[\Gamma^{\alpha}(q) + \Gamma^{\alpha}(-q)] \tag{5.37}$$

Yet this property of the Γ^{α} and their completeness (5.35c) are just the requirements for the uniqueness theorem of the projection operators $\mathcal{E}_k(\not{q}^2)$ to apply [see equations (5.13)–(5.15)]. Hence we have proved that

$$\mathcal{E}_k(\not{q}^2) = \sum_{m_{\alpha}=m_k} \Gamma^{\alpha}(q) + \Gamma^{\alpha}(-q) \tag{5.38}$$

where the sum is extended over all those states $\alpha = (s\rho)$ that have the same mass $m_\alpha = m_k$. This implies for the Klein–Gordon divisor

$$d(q) = (\mathbf{1} + \not{q}) \sum_\alpha c\mu_\alpha(1)\left[\Gamma^\alpha(q) + \Gamma^\alpha(-q)\right] \tag{5.39}$$

$c\mu_\alpha(1)$ being given by (5.26b); we have simply written μ_α for μ_{k_α}.

We shall need the values of the Klein–Gordon divisor at the various mass shells $q^2 = m_k^2$. From (5.26b) we deduce that

$$c\mu_j(1)\big|_{q^2 = m_k^2} = \delta_{jk} m_k^2 r_k^{-1} \tag{5.40}$$

with the constants r_k defined as

$$r_k^{-1} = \prod_{j \neq k} \left(-m_k^2 + m_j^2\right) \tag{5.41}$$

[These coefficients r_k are just the residua of $[\Pi(-q^2 + m_j^2)]^{-1}$ at $q^2 = m_k^2$. See equations (6.9) and (6.10) below.] For the momentum $q = {}_kp = (E_k, \mathbf{p}), E_k = {}_+(p^2 + m_k^2)^{1/2}$ we derive from (5.34) and (5.36) that

$$(\mathbf{1} + {}_k\not{p})\mathcal{E}_k({}_k\not{p}^2) = 2 \sum_{m_\alpha = m_k} \Gamma^\alpha({}_kp) = 2N_k^{-2} \sum_{m_\alpha = m_k} u(\mathbf{p}\alpha) \otimes \bar{u}(\mathbf{p}\alpha) \tag{5.42}$$

We find that on the mass shell

$$d({}_kp) = r_k^{-1} \sum_{m_\alpha = m_k} u(\mathbf{p}\alpha) \otimes \bar{u}(\mathbf{p}\alpha) \tag{5.43a}$$

Similarly we obtain

$$d(-{}_kp) = r_k^{-1} \sum_{m_\alpha = m_k} v(\mathbf{p}\alpha) \otimes \bar{v}(\mathbf{p}\alpha) \tag{5.43b}$$

This last equation also follows from the general relation

$$\beta_5 d(q) \beta_5 = d(-q) \tag{5.44}$$

as $d(q)$ is a simple polynomial in $\not{q}$. Analogously it is evident that under Lorentz transformations

$$D(g)\, d(q)\, D^{-1}(g) = d(q') \tag{5.45}$$

for arbitrary complex 4-vectors q^μ, and with $q' = L(g)q$. The completeness relation (4.19) now reads in terms of the Klein–Gordon divisor

$$\sum_{k=1}^{n} \frac{r_k}{2E_k(\mathbf{p})}\left[d(E_k, \mathbf{p}) - d(-E_k, \mathbf{p})\right] = \beta_0^{-1} \tag{5.46}$$

In the formulas (5.26a), (5.29), and (5.30) the singularity at $q^2=0$ is only apparent: The various terms contributing to the Klein–Gordon divisor conspire such that the appropriate cancellations occur. After all, according to (5.27) $d(q)$ is a simple matrix polynomial in q. From the formulas (5.27a) and (5.27b) we infer that the degree of the Klein–Gordon divisor as a polynomial in q is not larger than $2n-1$. This is also the actual degree of $d(q)$ as $2n-1$ is the smallest possible value consistent with the definition (5.1) of the Klein–Gordon divisor. For a certain class of wave equations Umezawa and Visconti (1956) have shown that the degree of the Klein–Gordon divisor is related to the maximum value of the spin contained in the wave equation. However, this result does not hold in general, and exceptions have already been noted before (Glass, 1971; Loide, 1977). In our case the degree of $d(q)$ depends solely on the number n of different masses m_k described by the wave equation, and is not directly related to the spins of the particles involved. No simple relation is to be expected, as even for a given type of wave equation, i.e., for a given transformation law $D(g)$, we can arrange degeneracies or remove them at will by varying the free linkage parameters b_{ik} contained in the β^μ. [For a simple example see Biritz (1975b).]

Here we shall be content with having proved that for every regular wave equation (1.5) there exists a corresponding Klein–Gordon divisor, without any additional requirements on the matrices β^μ or the transformation law $D(g)$. In another article we shall investigate the actual form of the Klein–Gordon divisor and the corresponding propagators in more detail.

6. THE INVARIANT FUNCTIONS

With the aid of the Klein–Gordon divisor it is now straightforward to obtain the various invariant functions of the wave equation (1.5) from the appropriate solutions of a set of Klein–Gordon equations. We prefer a uniform notation of these functions for Boson and Fermion fields as well, and we follow the sign conventions of Bogolubov et al. (1975). Hence certain of our invariant functions differ in sign from the definitions found, for example, in Schweber (1961) or Bjorken–Drell (1965).

We begin with the invariant solutions $\Delta_k(x)\equiv\Delta(x|m_k^2)$ and Green's functions belonging to a single mass m_k, i.e., the solutions of

$$(\square+m_k^2)\Delta_k=0 \tag{6.1a}$$

and the inhomogeneous equation

$$(\square+m_k^2)\Delta_{\text{inh}}=\delta^4(x) \tag{6.1b}$$

The various functions may be defined as contour integrals in the complex q^0 plane:

$$\Delta_k(x) = (2\pi)^{-4} \int_C d^4q e^{-iqx} / (-q^2 + m_k^2) \tag{6.2}$$

where the appropriate contours for the functions Δ, $\Delta^{\pm}$, Δ^R, Δ^A, and Δ^F are indicated in Figure 7. First there is the Jordan–Pauli commutator function

$$\Delta_k(x) = i(2\pi)^{-3} \int d^4q \varepsilon(q_0) \delta(q^2 - m_k^2) e^{-iq\cdot x} \tag{6.3}$$

This function has the special values

$$\Delta_k(0,\mathbf{r}) = 0 \tag{6.4a}$$

and

$$\frac{\partial \Delta_k}{\partial t}(0,\mathbf{r}) = \delta^3(\mathbf{r}) \tag{6.4b}$$

We define the positive and negative frequency functions $\Delta_k^{\pm}$ as

$$\Delta_k^{\pm}(x) = \pm i(2\pi)^{-3} \int \frac{d^3p}{2E_k(\mathbf{p})} \exp(\mp i\, {}_kp\cdot x) \tag{6.5a}$$

with ${}_kp = (E_k, \mathbf{p})$. Obviously we have

$$\Delta_k^{+}(-x) = -\Delta_k^{-}(x) \tag{6.5b}$$

The Green's functions Δ^R, Δ^A, and Δ^F can be built from the above

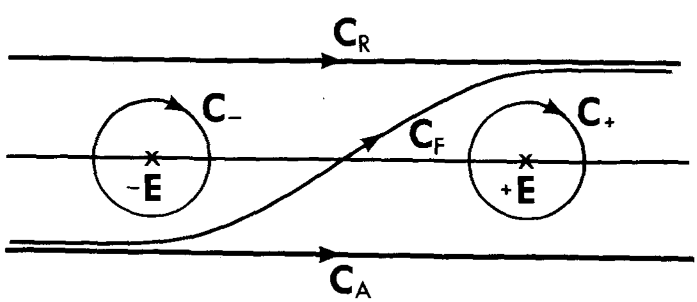


Fig. 7. Contours in the complex q_0 plane for the various functions defined in equation (6.2).

solutions Δ_k and $\Delta_k^{\pm}$ of the homogeneous equation and the step function $\theta(x) \equiv \theta(x_0)$ according to

$$\Delta_k^R(x) = \theta(x)\Delta_k(x) \tag{6.6a}$$

$$\Delta_k^A(x) = -\theta(-x)\Delta_k(x) \tag{6.6b}$$

$$\Delta_k^F(x) = \theta(x)\Delta_k^+(x) - \theta(-x)\Delta_k^-(x) \tag{6.6c}$$

From the properties of Δ_k and $\Delta_k^{\pm}$ it can be shown directly that the so-defined functions $\Delta^{R,A,F}$ satisfy the inhomogeneous equation (6.1b).

These functions are easily generalized for the multiple-mass equation

$$\prod_{k=1}^{n} (\Box + m_k^2)\Delta = 0 \tag{6.7a}$$

and the corresponding inhomogeneous equation

$$\prod_{k=1}^{n} (\Box + m_k^2)\Delta_{\text{inh}} = \delta^4(x) \tag{6.7b}$$

all the masses m_k being different from each other. We write $\Delta(x) \equiv \Delta(x|m_1^2 m_2^2 \cdots m_n^2)$ for the invariant functions associated with Eqs. (6.7a), and (6.7b); again we can express them as contour integrals:

$$\Delta(x) = (2\pi)^{-4} \int_C d^4q e^{-iq\cdot x} / \Pi(-q^2 + m_k^2) \tag{6.8}$$

The integrand has simple poles at $\pm E_k(\mathbf{q})$ in the complex q_0 plane, and the contours for the various invariant functions go around these singularities analogously to Figure 7. The invariant functions $\Delta(x)$ can be simply expressed as linear combinations of the functions $\Delta_k(x)$ belonging to one single mass m_k. For this purpose, and also to prove that certain time derivatives of $\Delta(x)$ vanish at $x_0 = 0$, we recall the identity (5.7) of the last section. Consider the polynomial

$$p(z) = (-z + z_1)(-z + z_2) \cdots (-z + z_n) \tag{6.9a}$$

with $z_k = E_k^2(\mathbf{q})$, and hence $z_i \neq z_k$ for $i \neq k$. Under this condition we have shown that

$$z^g = \sum_{k=1}^{n} z_k^g p_k(z) / p_k(z_k) \tag{6.9b}$$

for every integer $g=0,1,2,\ldots,n-1$. Introducing the constants r_k [see also equations (5.41) and (5.43)]

$$r_k^{-1}=p_k(z_k)=\prod_{j\neq k}\left(-E_k^2+E_j^2\right)$$

$$\equiv\prod_{j\neq k'}\left(-m_k^2+m_j^2\right) \tag{6.10}$$

we obtain for $g=0$

$$1=\sum_{k=1}^{n} r_k p_k(z) \tag{6.11a}$$

or

$$1/p(z)=\sum_{k=1}^{n} r_k/(-z+z_k) \tag{6.11b}$$

and in particular that

$$1/\Pi\left(-q^2+m_k^2\right)=\sum_{k=1}^{n} r_k/\left(-q^2+m_k^2\right) \tag{6.11c}$$

From the identity (6.9b) we can then derive various sum rules for the residua r_k. Assuming $g\neq 0$ in (6.9b) we obtain at $z=0$

$$0=\sum_{k=1}^{n} z_k^g r_k p_k(0)=p(0)\sum_{k=1}^{n} r_k z_k^{g-1} \tag{6.12a}$$

that is

$$\sum_{k=1}^{n} r_k z_k^j=0 \tag{6.12b}$$

for $j=0,1,2,\ldots,n-2$. Finally, setting $g=n-1$ and comparing the coefficients of z^{n-1} in (6.9b) we find

$$\sum_{k=1}^{n} r_k z_k^{n-1}=(-1)^{n-1} \tag{6.12c}$$

Hence we have the identities

$$\sum_{k=1}^{n} r_k E_k(\mathbf{q})^{2j}=0 \tag{6.13a}$$

for $j=0,1,2,\ldots,n-2$, and

$$\sum_{k=1}^{n} r_k E_k(\mathbf{q})^{2n-2}=(-1)^{n-1} \tag{6.13b}$$

for all values of $\mathbf{q}$.

The function $\Delta(x)$ can therefore be expanded as

$$\Delta(x)=\sum_{k=1}^{n} r_k \Delta_k(x) \tag{6.14}$$

and so can the positive- and negative-frequency solutions $\Delta^{\pm}(x)$ of the homogeneous equation (6.7a). Obviously $\Delta(x)$ has the same invariance and symmetry properties as the individual functions $\Delta_k(x)$. As expressed in the identity (6.13a), the various terms in $\Delta(x)$ conspire such that

$$\partial_0^j \Delta(x)=0 \qquad \text{at } x_0=0 \tag{6.15a}$$

for all integers $j=0,1,2,\ldots,2n-2$. [Of course, owing to symmetry the jth time derivative of $\Delta(x)$ vanishes at $x_0=0$ for all even j.] Furthermore, from (6.13b) we deduce

$$\partial_0^{2n-1}\Delta(x)=\delta^3(\mathbf{r}) \qquad \text{at } x_0=0 \tag{6.15b}$$

These above equations are the straightforward generalizations of the familiar relations (6.4a) and (6.4b) which hold for the single-mass ($n=1$) Klein–Gordon equation; for another special value of $\Delta(x)$ at $x_0=0$ see (6.23) below. From the definition (6.8) of the invariant functions as contour integrals it is obvious that identical expansions also hold for the Green's functions $\Delta^{R,A,F}$; for example

$$\Delta^R(x)=\sum_{k=1}^{n} r_k \Delta_k^R(x) \tag{6.16}$$

The so-defined function actually satisfies the inhomogeneous equation (6.7b). We find

$$\prod_{j=1}^{n}(\Box+m_j^2)\Delta^R(x)=\sum_{k=1}^{n} r_k\left[\prod_{j\neq k}(\Box+m_j^2)\right](\Box+m_k^2)\Delta_k^R(x)=\delta^4(x) \tag{6.17a}$$

as from (6.11c) there follows the identity

$$\sum_{k=1}^{n} r_k \prod_{j\neq k}(\Box+m_j^2)\equiv 1 \tag{6.17b}$$

Thus also for the multimass equation (6.7b) its Green's functions can be built from the step function $\theta(x)$ and the solutions $\Delta^{\pm}(x)$ of the homogeneous equation:

$$\Delta^{R}(x)=\theta(x)\Delta(x) \tag{6.18a}$$

$$\Delta^{A}(x)=-\theta(-x)\Delta(x) \tag{6.18b}$$

$$\Delta^{F}(x)=\theta(x)\Delta^{+}(x)-\theta(-x)\Delta^{-}(x) \tag{6.18c}$$

It is instructive to verify directly that these functions are solutions of the inhomogeneous equation (6.7b). From the fact that all the time derivatives $\partial_0^j\Delta(x)$ vanish at $x_0=0$ for $j=0,1,2,\ldots,2n-2$ we obtain the recursion formula

$$\partial_0^j\theta(x)\Delta(x)=\delta(x_0)\partial_0^{j-1}\Delta(x)+\theta(x)\partial_0^j\Delta(x) \tag{6.19a}$$

and hence that the time derivatives ∂_0^j can be moved past the step function without any contact terms:

$$\partial_0^j\theta(x)\Delta(x)=\theta(x)\partial_0^j\Delta(x) \tag{6.19b}$$

for $j=0,1,2,\ldots,2n-1$. Consider now, for example, equation (6.18a). According to (6.19b) the only nonvanishing contribution stems from the highest time derivative ∂_0^{2n},

$$\begin{aligned}\prod_{k=1}^{n}(\Box+m_k^2)\theta(x)\Delta(x)&=[\partial_0\theta(x)]\partial_0^{2n-1}\Delta(x)\\&=\delta(x_0)\partial_0^{2n-1}\Delta(x)=\delta^4(x)\end{aligned} \tag{6.20}$$

We finally come to the invariant solutions and Green's functions of the wave equation

$$(-i\partial\!\!\!/+\mathbf{1})S=0 \tag{6.21a}$$

and of the inhomogeneous equation

$$(-i\partial\!\!\!/+\mathbf{1})S_{\text{inh}}=\delta^4(x) \tag{6.21b}$$

In complete analogy to the Dirac equation these functions are obtained by applying the Klein–Gordon divisor $d(i\partial)$ onto the corresponding functions $\Delta(x)$ of the multimass equations (6.7a) or (6.7b). First there is the invariant function

$$S(x)=d(i\partial)\Delta(x) \tag{6.22a}$$

which satisfies (6.21a), and analogously defined positive- and negative-frequency solutions $S^{\pm}(x)$. In exactly the same way we obtain the Green's functions $S^{R,A,F}$; for example

$$S^R(x)=d(i\partial)\Delta^R(x) \tag{6.22b}$$

which satisfies the inhomogeneous equation (6.21b) and the appropriate boundary conditions.

From (5.46) [i.e., primarily from the completeness relation (4.19)] we deduce the special value

$$S(x_0=0,\mathbf{r})=d(i\partial)\Delta(x)|_{x_0=0}=i\beta_0^{-1}\delta^3(\mathbf{r}) \tag{6.23}$$

In Section 8 we shall find that $S(x)$ actually is the commutator function for the corresponding quantum field operators [see equation (8.19) below]. Therefore we obtain the remarkable and consequential result that the commutator of the field operators based on regular wave equations is not more singular at the apex of the light cone than in the familiar Dirac case, independent of the (arbitrarily high) spins of the particles involved!

This result has important consequences. At the purely technical level it implies that also for the wave equation (6.21b) its Green's functions can be constructed in the familiar fashion from the step function $\theta(x)$ and the solutions $S^{\pm}(x)$ of the homogeneous equation, i.e., that there are no contact terms. For example, the uniquely determined retarded Green's function $S^R(x)$ is defined as that solution of (6.21b) that obeys the boundary condition $S^R(x)=0$ for $x_0<0$. However, these are just the properties of the function $\theta(x)S(x)$: It trivially satisfies the boundary condition and according to (6.23) it is also a solution of (6.21b),

$$(-i\partial\!\!\!/+1)\theta(x)S(x)=-\beta_0\delta(x_0)S(x)+\theta(x)(-i\partial\!\!\!/+1)S(x)\equiv\delta^4(x) \tag{6.24}$$

Hence $S^R(x)-\theta(x)S(x)$ is a solution of the homogeneous wave equation which vanishes identically for $x_0<0$, and therefore

$$S^R(x)=\theta(x)S(x) \tag{6.25a}$$

By similar arguments we prove

$$S^A(x)=-\theta(-x)S(x) \tag{6.25b}$$

and

$$S^F(x)=\theta(x)S^+(x)-\theta(-x)S^-(x) \tag{6.25c}$$

Equations (6.18a), (6.22b), and (6.25a) imply the identity

$$d(i\partial)\theta(x)\Delta(x)=\theta(x)d(i\partial)\Delta(x) \tag{6.26}$$

i.e., that the Klein–Gordon divisor can be exchanged with the step function without any additional terms proportional to a $\delta(x_0)$ function and its derivatives; the other relations (6.25b) and (6.25c) then follow from (6.26) and the identity $\theta(x)+\theta(-x)=1$. However, equation (6.26) is a trivial consequence of the relations (6.19b) proved above and the fact that the Klein–Gordon divisor $d(i\partial)$ is a simple matrix polynomial in ∂^μ of degree $2n-1$. For theories in which (6.26) is not valid extra terms proportional to the $\delta(x_0)$ function and its derivatives appear in the propagator and destroy its Lorentz covariance. Noncovariant contact (that is, temporary local) terms have then to be added to the Hamiltonian in order to cancel these noncovariant terms in the propagator. The appearance of such contact terms has been a characteristic and embarrassing complication of almost any theory of higher spin considered up to now (see, e.g., Weinberg, 1964a, b; 1968; Umezawa and Visconti, 1956). However, this appears to be an artificial difficulty which is not inherent in the nature of higher spin fields, as we have found all regular wave equations, describing any number of particles with arbitrarily high spins, to be free of such contact terms. This is not merely a mathematical nicety as it is well known (see, e.g., Umezawa, 1956) that, within the framework of the Källén (1950) and Yang–Feldman (1950) formalism, the validity of (6.26) ensures the causality of the interacting fields. These and related topics will be fully discussed in another article devoted to the causality and stability of regular wave equations in external fields.

All these invariant functions transform under the Lorentz group according to

$$D(g)S(x)D^{-1}(g)=S(x') \tag{6.27}$$

$x'=L(g)x$. For the commutator functions we obtain the symmetry property under inversion

$$S^+(-x)=-\beta_5S^-(x)\beta_5 \tag{6.28a}$$

and consequently for the Green's functions

$$S^R(-x)=\beta_5S^A(x)\beta_5 \tag{6.28b}$$

$$S^F(-x)=\beta_5S^F(x)\beta_5 \tag{6.28c}$$

From the above functions we can easily obtain the corresponding invariant functions $\bar{S}(x)$ of the adjoint wave equation

$$\bar{S}(x)(i\overleftarrow{\partial\!\!\!/}+\mathbf{1})=0 \tag{6.29a}$$

and of the inhomogeneous equation

$$\bar{S}_{\text{inh}}(i\overleftarrow{\partial\!\!\!/}+\mathbf{1})=\delta^4(x) \tag{6.29b}$$

For example, we find

$$\bar{S}^R(x)=S^A(-x)=\beta_5 S^R(x)\beta_5 \tag{6.30}$$

From the theory of regular wave equations we gain uniquely determined propagators for particles of any spin, $S^F(x)=d(i\partial)\Delta^F(x)$, and in particular unambiguous expressions for these propagators off the mass shell. [See equations (5.29)–(5.34) and (5.39) above.] The explicit form of these propagators will be studied somewhere else; there we also want to discuss the various ad hoc prescriptions (Sugar and Sullivan, 1968; Steele, 1970; Jenkins, 1969, 1971, 1972b) that have been proposed, without the benefit of a consistent field theory of higher spin, to obtain propagators free of singularities.

7. *P*, *T*, AND *C*

Thus far we have developed the general theory of regular wave equations without assuming the validity of parity, time reversal, or charge conjugation, as these transformations are not exact symmetries of the laws of nature. In particular we have defined the adjoint wave function and the scalar product without the aid of the usual hermitizing matrix, i.e., without postulating the theory to be manifestly covariant under parity. Similarly, the existence of antiparticles already followed without invariance under charge conjugation. Here in this section we want to study the restrictions and simplifications due to these discrete symmetries.

In complete analogy to the local transformation property (1.2) or (3.25) of the wave function under the restricted Poincaré group we also postulate a local transformation law under P, T, and C. Beginning with parity, we assume the existence of a unitary operator T_P defined as

$$T_P\psi(x)=V\psi(\tilde{x}) \tag{7.1}$$

with $\tilde{x}=Px=(x_0,-\mathbf{r})$, and V being a nonsingular matrix acting on the components of the wave function. Assuming no superselection rules (Wick, Wightman, and Wigner, 1952) we normalize for simplicity

$$V^2 = 1 \tag{7.2}$$

For interacting fields a different normalization may be necessary (Racah, 1937; Yang and Tiomno, 1950; Feinberg and Weinberg, 1959)—this would only require minor modifications of our formulas below.

The parity transformation has to obey the group law (3.7b) with the elements of the restricted Poincaré group; we obtain the condition

$$V^{-1}D(g)V = D(g^{\dagger -1}) \tag{7.3}$$

Hence a local implementation of parity is only possible for a quite restricted class of wave equations: The transformation law $D(g)$ of the wave function under the Lorentz group has to be pseudounitary, i.e., $D(g)$ has to contain with every irreducible component $k=(A,B)$ also its conjugate representation $\bar{k}=(B,A)$. [In fact, we shall see below that from the nonsingularity of the parity matrix V there follows the even stronger result that the irreducible representation (A,B) and its conjugate (B,A) have to occur in $D(g)$ with the same multiplicity.] For the theory to be manifestly covariant under parity, with every solution $\psi(x)$ of the wave equation also its parity-transformed (7.1) has to be a solution of the same equation. This implies

$$V^{-1}\beta^{\mu}V = P^{\mu}{}_{\nu}\beta^{\nu} \tag{7.4}$$

in complete analogy to (1.3a) for restricted Lorentz transformations. In particular V commutes with β^0, and we may therefore assume the eigenvectors $w(\alpha\varepsilon)$ of β^0, equation (4.6b), also to be eigenstates of the parity matrix,

$$Vw(\alpha\varepsilon) = \eta^P_{\alpha\varepsilon} w(\alpha\varepsilon) \tag{7.5a}$$

with the intrinsic parities $\eta = \pm 1$. It then follows from the orthogonality and completeness relations (4.7a) and (4.7b) that the adjoint spinor $\bar{w}(\alpha\varepsilon)$ is an eigenvector of V^{-1} with the same eigenvalue,

$$\bar{w}(\alpha\varepsilon)V^{-1} = \eta^P_{\alpha\varepsilon}\bar{w}(\alpha\varepsilon) \tag{7.5b}$$

The effect of the parity transformation T_P on the plane wave solutions $f(x|p\alpha\varepsilon)$ is easily derived. We find

$$T_P f(x|ps\sigma\rho\varepsilon) = \eta^P_{\alpha\varepsilon} f(x|\tilde{p}s\sigma'\rho\varepsilon)\,D^s(r)_{\sigma'\sigma} \tag{7.6a}$$

with the rotation $r=[\tilde{p}\alpha]^{\dagger}[p\alpha]$, as already derived in (3.8b) from general invariance arguments. To simplify the notation we shall suppress the magnetic quantum numbers σ and σ' in the following; the above equation is then written as

$$T_P f(x|p\alpha\varepsilon)=\eta^P_{\alpha\varepsilon} f(x|\tilde{p}\alpha\varepsilon)\,D^s(r) \tag{7.6b}$$

According to (4.22) the corresponding adjoint wave function is given by

$$(\overline{T_P f})(x|p\alpha\varepsilon)=\eta^P_{\alpha\varepsilon} D^s(r^{-1})\bar{f}(x|\tilde{p}\alpha\varepsilon) \tag{7.6c}$$

which, however, is just the same as $\bar{f}(\tilde{x}|p\alpha\varepsilon)V^{-1}$. Hence we obtain in general for the parity transformed adjoint wave function

$$(\overline{T_P\psi})(x)=\bar{\psi}(\tilde{x})V^{-1} \tag{7.7}$$

This ensures that T_P is a unitary operator within our scalar product (4.23).

To make further progress we need a more explicit expression for the parity matrix V. According to (7.3) V relates $D(g)$ to $D(g^{\dagger-1})$. For a single irreducible representation D^{AB} we observe that $D^{AB}(g^{\dagger-1})$ merely differs by the order of the rows and columns from the representation $D^{BA}(g)$; the appropriate rearrangement can be accomplished by a unitary transformation. For this purpose we define the $(2A_k+1)(2B_k+1)$-dimensional matrix $(U_{k\bar{k}})$ by its elements

$$(U_{k\bar{k}})_{ab;\bar{b}\bar{a}}=\delta_{a\bar{a}}\delta_{b\bar{b}} \tag{7.8a}$$

evidently

$$(U_{k\bar{k}})(U_{\bar{k}k})=\mathbf{1}_k \tag{7.8b}$$

These matrices connect the irreducible representations k and $\bar{k}$,

$$D^k(g^{\dagger-1})=(U_{k\bar{k}})D^{\bar{k}}(g)(U_{\bar{k}k}) \tag{7.9}$$

To express the parity matrix in terms of the $(U_{k\bar{k}})$ we subdivide V into blocks of matrix elements (V_{ik}), in accordance with our completely reduced standard form of $D(g)$, introduced in Section 3, and with the analogous block form (β^{μ}_{ik}) of the matrices β^{μ}. Considering the i,k block of the equation $VD(g^{\dagger-1})=D(g)V$ we learn from Schur's lemma that

$$(V_{ik})=\delta(\bar{i},k)(-1)^{A_i+B_i-F}v_{ik}(U_{i\bar{i}}) \tag{7.10}$$

There the symbol $\delta(i,j)$, which has to be distinguished from the ordinary

Kronecker delta δ_{ij}, is defined as

$$\delta(i,j)=\delta_{A_iA_j}\delta_{B_iB_j} \tag{7.11}$$

i.e., it indicates that the representations D^i and D^j are identical. [We recall that a given irreducible representation may occur more than once in the decomposition of $D(g)$.] The v_{ik} are arbitrary constants, and the phase factor has been chosen to obtain a simple expression for the reduced parity matrix $\mathcal{V}(s)$, equation (7.15) below. There F stands for $F=0$ in the case of bosons, whereas $F=\frac{1}{2}$ for fermions. For simplicity we are assuming that there are no superselection rules; hence the wave equation describes either bosons or fermions.

From (7.10) it is evident that the only nonvanishing elements of the parity matrix V occur in those blocks (V_{ik}) that connect a representation (A,B) with its conjugate (B,A). This implies that

$$\begin{aligned}(\beta_5 V\beta_5)_{ik}&=(-1)^{2A_i+2A_k}(V_{ik})=(-1)^{2A_i+2B_i}(V_{ik})\\ &\equiv(-1)^{2s}(V_{ik})\end{aligned} \tag{7.12a}$$

and therefore

$$\beta_5 V\beta_5=(-1)^{2s}V \tag{7.12b}$$

From this last equation we obtain the well-known result that the intrinsic parities $\eta_{\alpha+}$ and $\eta_{\alpha-}$ of particles and antiparticles are related according to

$$\eta^P_{\alpha+}\eta^P_{\alpha-}=(-1)^{2s} \tag{7.13}$$

We now want to show that in a theory that is manifestly covariant under parity the number of times a certain irreducible representation $(A,B), D^i=D^{AB}$ for $i=1,2,\ldots,n$ is contained in the transformation law $D(g)$ has to be equal to the multiplicity with which the conjugate representation (B,A) occurs in $D, D^{\bar{i}}=D^{BA}$ for $\bar{i}=\bar{1},\bar{2},\ldots,\bar{n}$; we may assume $n\geqslant\bar{n}$. Consider the n linearly independent column vectors $c(i)$, the only nonvanishing components of which are in the block corresponding to the representation $D^i, c_k(i)=\delta_{ik}\chi_i(s\sigma)$, $i=1,2,\ldots,n$. There $\chi_i(s\sigma)$ are the spin projectors introduced in (2.2), and we have labeled the components of $c(i)$ appropriately to the block form of $D(g)$ and V, as discussed in Section 3 in connection with equation (3.18b). Applying now the parity matrix V, equation (7.10), onto these n linearly independent vectors $c(i)$, the ensuing vectors $Vc(i)$ can be expressed as linear combinations of the $\bar{n}$ vectors $c(\bar{k}), n\geqslant\bar{n}$. However, V being nonsingular demands $n=\bar{n}$.

For the effect of the parity matrix on the spinors $w(\alpha\varepsilon)$ we note that according to the symmetry properties of the Clebsch–Gordon coefficients (2.2a)

$$(U_{k\bar{k}})\chi_{\bar{k}}(s\sigma)=(-1)^{A_k+B_k-s}\chi_k(s\sigma) \tag{7.14a}$$

To determine the intrinsic parities of the various spinors $w(\alpha\varepsilon)$ we only have to know the effect of the parity transformation on the reduced wave functions. We write

$$[Vw(\alpha\varepsilon)]_i=z_i^P(\alpha\varepsilon)\chi_i(s\sigma) \tag{7.14b}$$

with

$$z_i^P(\alpha\varepsilon)=\sum_l \mathcal{V}(s)_{il}z_l(\alpha\varepsilon) \tag{7.14c}$$

We obtain for the so-defined N-dimensional reduced parity matrix

$$\mathcal{V}(s)_{ik}=(-1)^{s-F}\delta(i,\bar{k})v_{ik} \tag{7.15a}$$

its only nonvanishing matrix elements connecting mutually conjugate representations. [Note the meaning of the symbol $\delta(i,j)$, equation (7.11).] We learn that the spin dependence of the reduced parity matrix can be completely factorized,

$$\mathcal{V}(s)=(-1)^{s-F}\mathcal{V} \tag{7.15b}$$

This reduced parity matrix satisfies

$$\mathcal{V}^2=\mathbf{1} \tag{7.16}$$

and it commutes with the reduced mass matrix (2.5),

$$\mathcal{V}\Lambda(s)=\Lambda(s)\mathcal{V} \tag{7.17a}$$

This last equation (viz., $V\beta^0=\beta^0V$) and (7.3) then imply the general relation $V^{-1}\beta^\mu V=(P\beta)^\mu$ for all μ. Equation (7.17a) is ensured for all spin values once it is satisfied for one particular s; in fact, using the explicit expressions (7.15) and (2.5) for the reduced matrices $\mathcal{V}$ and $\Lambda(s)$, we obtain the following condition on the v_{ik} and the linkage parameters b_{ik}:

$$\sum_l v_{il}b_{lk}\delta(l,\bar{i})=(-1)^{\varphi_{ik}}\sum_l b_{il}v_{lk}\delta(l,\bar{k}) \tag{7.17b}$$

with the phase $\varphi_{ik}=A_i+B_i+A_k+B_k+2F+1$.

These above results allow a quite detailed discussion of the intrinsic parities η_α of any parity covariant wave equation (1.5). According to (7.17a) we can find simultaneous eigenvectors of the N-dimensional reduced matrices $\Lambda(s)$ and $\mathcal{V}$,

$$\Lambda(s)z(s\rho)=\lambda_\rho(s)z(s\rho) \tag{7.18a}$$

$$\mathcal{V}z(s\rho)=\eta_\rho z(s\rho) \tag{7.18b}$$

Within a given branch $\lambda_\rho(s)$ [see the remarks following equation (3.21)] the eigenvalue η_ρ of $\mathcal{V}$ is independent of the spin, as from (7.16) the only possible values of η_ρ are ± 1, and among them no continuous transition is possible. The intrinsic parities are given by $\eta_\rho(-1)^{s-F}$. For bosons the parities of particles and antiparticles are the same, and therefore within the corresponding branch of the mass spectrum the intrinsic parities $\eta_{\alpha+}$ always alternate according to

$$\eta^P_{\alpha+}=\eta_\rho(-1)^s \qquad \text{for bosons} \tag{7.19a}$$

irrespective of any singularities in the mass spectrum. For fermions this alternating sequence of parities along a given branch is interrupted at every infinity in the mass spectrum,

$$\eta^P_{\alpha+}=\pm\eta_\rho(-1)^{s-1/2} \qquad \text{for fermions} \tag{7.19b}$$

the particles immediately to the right and left of the singularity (at unphysical values of s) having the same intrinsic parity. We present a schematic illustration of mass parity spectra in Figure 8. The intrinsic parities η^P_α of all the particles are uniquely determined once the overall phases η_ρ for the various branches of the mass spectrum are known. At least theoretically, these $\eta_\rho=\pm 1$ can be chosen arbitrarily. The different choices for the η_ρ reflect the existence of several physically inequivalent solutions of (7.17). Prescribing the η_ρ determines the phases of the linkage parameters b_{ik} and therefore also affects, sometimes rather drastically, the shape of the mass spectrum; for a simple example see (Biritz, 1975b).

Manifest covariance under parity is only possible for a pseudounitary transformation law $D(g)$. For arbitrary wave equations (1.1) this in general can only be achieved by a doubling of the number of components of the wave function. We emphasize that this doubling of the degrees of freedom (and hence of the number of particles described by the wave equation) does not necessarily imply the existence of degenerate parity doublets, i.e., doublets of particles having the same mass but opposite intrinsic parities. For wave equations containing spin zero or spin one-half particles in their

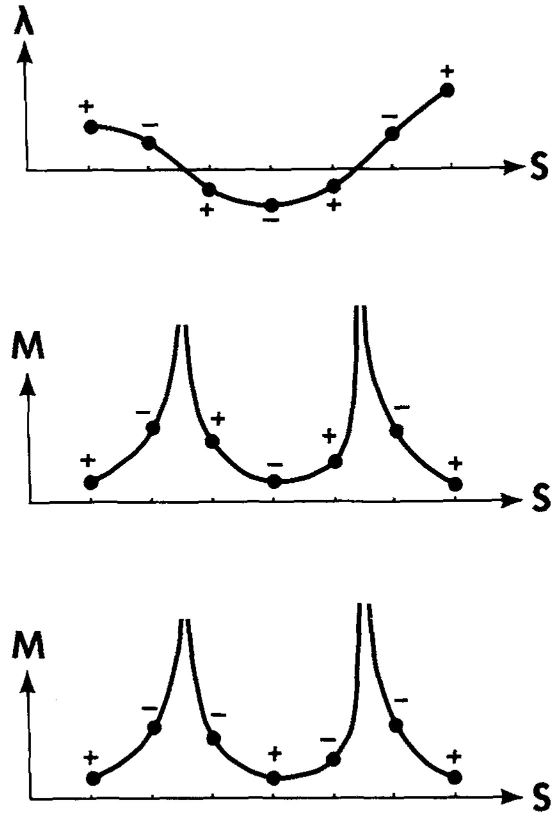


Fig. 8. Sketch of typical spin-parity values for one particular branch of the mass spectrum. The graph on top depicts $\lambda(s)$, which may assume both positive and negative values (particles and antiparticles). The parities of the corresponding states alternate according to equation (7.15b). In the two graphs below we have plotted the masses of the corresponding particles, $m(s)=|\lambda(s)|^{-1}$. From the connection between the intrinsic parities of particles and antiparticles we obtain the spin-parity values for bosons (shown in the middle) and fermions (bottom).

spectrum such degenerate parity doublets occur only for special values of the linkage parameters b_{ik} but otherwise we have much freedom in the way we can split these doublets and even change their relative parities. [For more details we again refer to Biritz (1975b).] Hence we see no compelling reason for a nonlocal implementation of parity as advocated in Hurley (1974).

The importance of parity (or of a hermitizing matrix) for the general theoretical framework seems to have been greatly overemphasized in the literature. There is the widely held belief that an invariant scalar product and a Lagrangian only exist for wave equations with a pseudounitary transformation law $D(g)$; there even is a formal proof for this to be a necessary condition (Gel'fand et al., 1963; Naimark, 1964). As a consequence one usually takes without any further discussion the parity (hermitizing) matrix as metric operator in the definition of the scalar product. This is an unfortunate choice which invariably is bound to cause trouble. For the sake of the argument assume β^0 to be Hermitian, as is usually done. From Lorentz covariance the $\boldsymbol{\beta}$ are then anti-Hermitian, and V simply coincides with the hermitizing matrix, $(\beta^\mu)^\dagger = (P\beta)^\mu = V^{-1}\beta^\mu V$. In that case we find

$$f(x|p\alpha\varepsilon)^\dagger V = \eta^P_{\alpha\varepsilon}\bar{f}(x|p\alpha\varepsilon) \tag{7.20}$$

$\bar{f}$ being our adjoint wave function as defined in Section 4. We have just learned from equations (7.19a) and (7.19b) that in general the intrinsic parities alternate within the various branches of the mass spectrum; hence the use of V as metric operator will in general lead to an indefinite metric (even in Fock space). The above-mentioned proof that only pseudounitary $D(g)$ render possible an invariant scalar product depends on the tacit assumption that the adjoint wave function $\bar{\psi}(x)$ is locally related to the complex-conjugate and transpose wave function $\psi^\dagger(x)$. It is perhaps tempting to demand such a local connection between $\bar{\psi}$ and $\psi^\dagger$, considering the postulated local transformation law of the wave function under the Poincaré group and P, T, and C. However, such an assumption is too restrictive and there is no fundamental reason for this hypothesis. In fact, we have shown in Section 4 how to define, for any wave equation (1.5), the adjoint wave function and an invariant scalar product with the usual properties (4.26)–(4.29); in the next section it will be seen that the ensuing metric in Fock space is positive definite.

For time reversal we require the existence of an antilinear operator A_T defined as

$$A_T\psi(x) = W\psi^*(Tx) \tag{7.21}$$

with $Tx = (-x_0, \mathbf{r})$, and W a nonsingular matrix acting on the components of the complex conjugate wave function. Assuming no superselection rules we normalize $A_T^2 = \varepsilon_T \mathbf{1}$ in accordance with (3.13a), i.e.,

$$WW^* = \varepsilon_T \mathbf{1} \tag{7.22}$$

$\varepsilon_T = \pm 1$ being a real phase factor. From the group law (3.13b) relating time reversal and the elements of the restricted Poincaré group we obtain the condition

$$W^{-1}D(g)W = D(g^{T-1}) \tag{7.23}$$

Hence a local implementation of time reversal does not impose any restrictions on the transformation law $D(g)$ as it is well known that for all Lorentz transformations $g \in SL(2,C)$

$$g^{T-1} = CgC^{-1} \tag{7.24}$$

with the matrix $C = C^{1/2} = -i\sigma_y$ as introduced in (3.11). For manifest covariance under time reversal, with every solution $\psi(x)$ of the wave equation (1.5) also its time-reversed (7.21) has to be a solution of the same equation. This implies for the matrices β^μ that

$$W^{-1}\beta^\mu W = P^\mu{}_\nu(\beta^\nu)^* \tag{7.25}$$

the asterisk again denoting the complex conjugate. In particular we find $\beta^0 W = W(\beta^0)^*$; consequently $Ww^*(s\sigma\rho\varepsilon)$ and $w(s\sigma'\rho\varepsilon)C^s_{\sigma'\sigma}$ are eigenvectors of β^0 belonging to the same eigenvalue which transform identically under rotations, using (3.11b). For simplicity we assume that there are no degeneracies (internal symmetries), and therefore

$$Ww^*(\alpha\varepsilon) = \eta^T_{\alpha\varepsilon} w(\alpha\varepsilon)C^s \tag{7.26}$$

where we have suppressed the indices σ and σ'. Iterating this last equation we find $\eta\eta^*\varepsilon_T(-1)^{2s} = 1$; therefore

$$\varepsilon_T = (-1)^{2s} \tag{7.27}$$

as already derived in (3.16) from general invariance arguments, and the $\eta^T_{\alpha\varepsilon}$ are (unobservable) phase factors of modulus one. Using the orthogonality and completeness relations of the spinors $w(\alpha\varepsilon)$ we obtain

$$\overline{w}^*(\alpha\varepsilon)W^{-1} = \eta^{T*}_{\alpha\varepsilon} C^{-s}\overline{w}(\alpha\varepsilon) \tag{7.28}$$

there we have written C^{-s} for the inverse of the matrix C^s.

On the plane wave solutions $f(x|p\alpha\varepsilon)$ the time reversal operation (7.21) has the required effect

$$A_T f(x|p\alpha\varepsilon) = \eta^T_{\alpha\varepsilon} f(x|\tilde{p}\alpha\varepsilon)D^s(r') \tag{7.29a}$$

with the rotation $r' = [\tilde{p}\alpha]^\dagger[p\alpha]C$ [see equation (3.14b)]. According to (4.22) the corresponding adjoint wave function is given by

$$(\overline{A_T f})(x|p\alpha\varepsilon) = \eta_{\alpha\varepsilon}^{T*} D^s(r')^{-1} \bar{f}(x|\tilde{p}\alpha\varepsilon) \tag{7.29b}$$

which is just the same as $\bar{f}(Tx|p\alpha\varepsilon)^* W^{-1}$. We thus find for the transformation law of the general adjoint wave function under time reversal

$$(\overline{A_T\psi})(x) = \bar{\psi}(Tx)^* W^{-1} \tag{7.30}$$

Together with (7.21) and (7.25) this last equation ensures that A_T is an antiunitary operator within our scalar product (4.23).

The explicit expression for the time reversal matrix W is easily obtained in our standardization of $D(g)$ and the matrices β^μ. From (7.23) and (7.24) and Schur's lemma the i,k block of matrix elements is found to be given, up to a constant factor w_{ik}, by the matrix $C^k \equiv C^{A_k} \otimes C^{B_k}$ [see also (3.11)]:

$$(W_{ik}) = w_{ik}\delta(i,k) C^k \tag{7.31}$$

the symbol δ (i,k) has been explained in (7.11) above. The only nonvanishing elements of the time reversal matrix W are in those blocks (W_{ik}) that connect identical representations $(A_i, B_i) = (A_k, B_k)$. Hence

$$(\beta_5 W \beta_5)_{ik} = (-1)^{2A_i + 2A_k} (W_{ik}) = (W_{ik}) \tag{7.32a}$$

i.e., β_5 commutes with W,

$$\beta_5 W = W \beta_5 \tag{7.32b}$$

From this we learn that the phase factors $\eta_{\alpha\varepsilon}^T$ for particles and antiparticles are identical,

$$\eta_{\alpha\varepsilon}^T = \eta_{\alpha,-\varepsilon}^T = \eta_\alpha^T \tag{7.33}$$

We write for the effect of the time reversal matrix W on the spinors $w(\alpha\varepsilon)$

$$(Ww(\alpha\varepsilon))_i = z_i^w(\alpha\varepsilon)\chi_i(s\sigma') C_{\sigma'\sigma}^s \tag{7.34a}$$

with the reduced wave functions

$$z_i^w(\alpha\varepsilon) = \sum_k \mathfrak{W}_{ik} z_k(\alpha\varepsilon) \tag{7.34b}$$

where we have introduced the elements $\mathfrak{W}_{ik}$ of the N-dimensional reduced time reversal matrix $\mathcal{W}$:

$$\mathcal{W}_{ik} = w_{ik}\delta(i,k) \tag{7.34c}$$

Equation (7.34b) only describes the effect of the matrix W on the reduced wave functions; the fully time-reversed wave functions are, of course, given by

$$z_i^T(\alpha\varepsilon) = \sum_k \mathcal{W}_{ik} z_k^*(\alpha\varepsilon) \tag{7.34d}$$

These reduced wave functions $z(s\rho)$ corresponding to particles at rest are determined by the equations

$$\Lambda(s)z(s\rho) = \lambda_\rho(s)z(s\rho) \tag{7.35a}$$

$$\mathcal{W}z^*(s\rho) = \eta_\rho^T(s)z(s\rho) \tag{7.35b}$$

From (7.22) and (7.34a) we learn that

$$\mathcal{W}\mathcal{W}^* = \mathbf{1}_N \tag{7.36}$$

$\mathbf{1}_N$ being the N-dimensional unit matrix. $\mathcal{W}$ satisfies with the reduced mass matrix $\Lambda(s)$ of (2.5)

$$\Lambda(s)\mathcal{W} = \mathcal{W}\Lambda^*(s) \tag{7.37}$$

This last equation (viz., $\beta_0 W = W\beta_0^*$) and the property (7.23) imply the general formula $W^{-1}\beta^\mu W = P_\nu^\mu(\beta^\nu)^*$ for all μ; we obtain the following relation between the linkage parameters b_{ik} and the matrix elements w_{ik} of $\mathcal{W}$:

$$\sum_l b_{il} w_{lk}\delta(l,k) = \sum_l w_{il}\delta(i,l) b_{lk}^* \tag{7.38}$$

Every invariant wave equation (1.5) automatically contains a negative-frequency solution corresponding to each solution with positive frequency. In the usual definition of charge conjugation one makes the additional demand of a local connection between particle and antiparticle solutions. Observing that the plane wave functions $f_\pm$ are proportional to $\exp(\mp ip\cdot x)$, we are led to define charge conjugation by the operation (antilinear in the first-quantized theory)

$$A_C\psi(x) = Z\psi^*(x) \tag{7.39}$$

There Z is a nonsingular matrix which acts on the components of the complex conjugate wave function. We normalize $A_C^2 = \varepsilon_C \mathbf{1}, \varepsilon_C = \pm 1$ being a real phase factor; hence

$$ZZ^* = \varepsilon_C \mathbf{1} \tag{7.40}$$

Charge conjugation is assumed to commute with all the elements of the Poincaré group,

$$T_{a,g} A_C = A_C T_{a,g} \tag{7.41a}$$

which implies

$$Z^{-1} D(g) Z = D(g)^* \tag{7.41b}$$

i.e., a pseudounitary transformation law $D(g)$. For a theory to be manifestly covariant under charge conjugation, with every solution $\psi(x)$ of the wave equation (1.5) also its charge conjugate (7.39) has to be a solution of the same wave equation. For this the matrices β^μ have to satisfy the condition

$$Z^{-1} \beta_\mu Z = -\beta_\mu^* \tag{7.42}$$

In particular we obtain $\beta_0 Z = -Z\beta_0^*$; thus $Zw^*(s\sigma\rho\varepsilon)$ and $w(s\sigma'\rho, -\varepsilon)C^s_{\sigma'\sigma}$ are eigenvectors of β^0 to the same eigenvalue which transform identically under rotations. Assuming no degeneracies we deduce

$$Zw^*(\alpha\varepsilon) = \eta_{\alpha\varepsilon}^{C*} w(\alpha, -\varepsilon) C^s \tag{7.43}$$

As expected, the matrix Z connects particle and antiparticle spinors. Iterating this last equation we find $(-1)^{2s} \eta_{\alpha,-\varepsilon} \eta_{\alpha\varepsilon}^* = \varepsilon_C$; anticipating the result of equation (7.51) below we infer

$$\varepsilon_C = +1 \tag{7.44}$$

and that the η^C are phase factors of modulus one. With the aid of the orthogonality and completeness relations (4.7a) and (4.7b) we obtain for the effect of the charge conjugation matrix on the adjoint spinors

$$\bar{w}^*(\alpha\varepsilon) Z^{-1} = \eta_{\alpha\varepsilon}^C C^{-s} \bar{w}(\alpha, -\varepsilon) \tag{7.45}$$

again we have denoted the inverse of the matrix C^s by C^{-s}. The plane wave solutions $f(x|p\alpha\varepsilon)$ are found to transform under charge conjugation as

$$A_C f(x|p\alpha\varepsilon) = \eta_{\alpha\varepsilon}^{C*} f(x|p\alpha, -\varepsilon) C^s \tag{7.46}$$

According to (4.22a) and (4.22b) the corresponding adjoint wave functions are then given by

$$(\overline{A_C f})(x|p\alpha\varepsilon) = \eta^C_{\alpha\varepsilon} C^{-s} \bar{f}(x|p\alpha, -\varepsilon) \tag{7.47}$$

this, however, is exactly the same as $\bar{f}(x|p\alpha\varepsilon)^* Z^{-1}$. Therefore we obtain the general transformation law of the adjoint wave function $\bar{\psi}(x)$ under charge conjugation

$$(\overline{A_C \psi})(x) = \bar{\psi}(x)^* Z^{-1} \tag{7.48}$$

which ensures A_C to be an antiunitary operator. [To be exact, we find $(A_C\psi, A_C\Phi) = -(\psi, \Phi)^*$, the scalar product (4.23) being indefinite in the first-quantized theory.]

From (7.41b) and Schur's lemma we derive the following explicit expression for the block of matrix elements (Z_{ik}) of the charge conjugation matrix Z:

$$(Z_{ik}) = z_{ik}\delta(i, \bar{k})(U_{\bar{k}k})C^k \tag{7.49}$$

There the z_{ik} are constant numerical factors [restricted by (7.42); see equation (7.55) below]; $(U_{\bar{k}k})$ is the similarity transformation which connects the irreducible representations $D^k(g)$ and $D^{\bar{k}}(g^{\dagger-1})$, as introduced in (7.9) above, and $C^k = C^{A_k} \otimes C^{B_k}$. The only vanishing matrix elements of Z are found in those blocks (Z_{ik}) that connect a given representation (A,B) with its conjugate (B,A). In particular, this implies for the matrix β_5

$$(\beta_5 Z \beta_5)_{ik} = (-1)^{2A_i + 2A_k}(Z_{ik}) = (-1)^{2A_i + 2B_i}(Z_{ik}) \tag{7.50a}$$

i.e., that

$$\beta_5 Z \beta_5 = (-1)^{2s} Z \tag{7.50b}$$

Thus the phase factors η^C for particles and antiparticles are related by

$$\eta^C_{\alpha, -\varepsilon} = (-1)^{2s} \eta^C_{\alpha\varepsilon} \tag{7.51}$$

The effect of the charge conjugation matrix Z on the spinors $w(\alpha\varepsilon)$ can be written in the form

$$(Zw(\alpha\varepsilon))_i = z'_i(\alpha\varepsilon)\chi_i(s\sigma')C^s_{\sigma'\sigma} \tag{7.52a}$$

The reduced wave functions $z'_i(\alpha\varepsilon)$ are given by

$$z'_i(\alpha\varepsilon) = \sum_k \mathcal{Z}_{ik}(s) z_k(\alpha\varepsilon) \tag{7.52b}$$

where we have introduced the N-dimensional reduced charge conjugation matrix $\mathfrak{Z}(s)$,

$$\mathfrak{Z}_{ik}(s) = (-1)^{s-F}\mathfrak{Z}_{ik} \tag{7.52c}$$

with

$$\mathfrak{Z}_{ik} = (-1)^{A_i + B_i - F} z_{ik}\delta(i, \bar{k}) \tag{7.52d}$$

$F = O(\frac{1}{2})$ for bosons(fermions). This reduced charge conjugation matrix satisfies

$$\mathfrak{Z}\mathfrak{Z}^* = (-1)^{2s}\mathbf{1}_N \tag{7.53}$$

and obeys with the reduced mass matrix $\Lambda(s)$ of (2.5) the relation

$$\Lambda(s)\mathfrak{Z} = -\mathfrak{Z}\Lambda^*(s) \tag{7.54}$$

Again this last equation (viz., $\beta_0 Z = -Z\beta_0^*$) and (7.41b) entail the validity of $Z^{-1}\beta_\mu Z = -\beta_\mu^*$ for all μ. We obtain the following condition on the linkage parameters b_{ik} and the constants z_{ik} appearing in the charge conjugation matrix Z:

$$\sum_l b_{il} z_{lk}\delta(l, \bar{k}) = -\sum_l z_{il}\delta(i, \bar{l}) b_{lk}^* \tag{7.55}$$

The N-component reduced wave functions $z(\alpha\varepsilon)$ describing particles at rest are then determined from the equations

$$\Lambda(s) z(s\rho) = \lambda_\rho(s) z(s\rho) \tag{7.56a}$$

and

$$\mathfrak{Z}(s) z^*(s\rho) = \eta_\rho^C(s)^* z(s\rho) \tag{7.56b}$$

Having developed the individual symmetry operations P, T, and C, we now could investigate their various products and phase relations. As these questions are of no great importance to our argument, and as they can be treated along standard lines (Bogolubov et al., 1975; Carruthers, 1971), we do not want to pursue this matter any further here. We close this section with the behavior of the Klein–Gordon divisor $d(q)$ under these discrete symmetries:

$$V^{-1} d(q) V = d(\tilde{q}) \tag{7.57a}$$

$$W^{-1} d(q) W = d(\tilde{q}^*)^* \tag{7.57b}$$

$$Z^{-1} d(q) Z = d(-q^*)^* \tag{7.57c}$$

There $q=(q^0,\mathbf{q})$ is an arbitrary complex 4-vector, $\tilde{q}=Pq=(q^0,-\mathbf{q})$, and the asterisk denotes the complex conjugate. These relations follow at once from the properties of the matrices β^μ and the fact that the Klein–Gordon divisor is a simple polynomial in $\not{q}=q_\mu\beta^\mu$ and $q^2=q_\mu q^\mu$, with otherwise real coefficients.

8. FREE QUANTUM FIELDS

In the presence of interaction there is no consistent physical interpretation of relativistic wave equations within the framework of a single-particle theory, and the main reason for our interest in regular wave equations lies in the corresponding quantum field theories. For the construction of field operators based on regular wave equations various levels of mathematical sophistication are possible. Here we shall follow the usual intuitive approach (Schweber, 1961; Pauli, 1973) of expanding the field in terms of a complete set of (plane wave) solutions of the wave equation and reinterpreting the amplitudes as creation or annihilation operators. All our loose talk about field operators $\psi(x)$ at a space–time point [which, strictly speaking, do not exist (Wightman, 1964)] can be given precise meaning with the aid of appropriately smeared fields $\psi(f)$ (Wightman, 1973). However, here we see no point in a more precise but perhaps less familiar language for such a relatively trivial topic as the free fields to be studied in this section.

The creation operator $a^\dagger(p\alpha)$ is defined to produce the single-particle state $|p\alpha\rangle$ when applied to the Poincaré invariant vacuum $|0\rangle$,

$$|p\alpha\rangle=a^\dagger(p\alpha)|0\rangle \tag{8.1}$$

From the transformation properties of these states under the Lorentz group we deduce for the creation and annihilation operators

$$U(g)a(p\alpha)U^{-1}(g)=D^s(W^{-1}(g,p\alpha))_{\sigma\sigma'}a(p'\alpha') \tag{8.2a}$$

$$U(g)\left[C^s_{\tau\sigma}a^\dagger(p\alpha)\right]U^{-1}(g)=D^s(W^{-1}(g,p\alpha))_{\tau\tau'}\left[C^s_{\tau'\sigma'}a^\dagger(p'\alpha')\right] \tag{8.2b}$$

with a summation over double indices. We usually do not need to write out the triplet of single-particle labels $\alpha=(s\sigma\rho)$ and $\alpha'=(s\sigma'\rho)$; furthermore, it should be clear from the context whether α' stands for $(s'\sigma'\rho')$ or $(s\sigma'\rho)$. $W(g,p\alpha)$ is the Wigner rotation defined in (3.5c), and the matrix C^s, equation (3.11), relates the representation $D^s(r)$ with its complex conjugate. These creation and annihilation operators are assumed to satisfy the canonical (anti)commutation relations

$$[a(p\alpha),a^{\dagger}(p'\alpha')]_{\pm}=2E_{\alpha}(\mathbf{p})\delta(\mathbf{p}-\mathbf{p}')\delta(\alpha,\alpha') \tag{8.3}$$

where this time $\alpha'=(s'\sigma'\rho')$. The metric in Fock space is positive definite, $E_{\alpha}(\mathbf{p})={}_{+}(\mathbf{p}^2+m_{\alpha}^2)^{1/2}$, and we shall see below that the microcausality condition demands the usual connection between spin and statistics.

We define the field operator $\psi(x)$ by its expansion in terms of the complete set of plane wave solutions $f_{\pm}(x|p\alpha)$ of the wave equation we derived in Section 3:

$$\psi(x)=(2\pi)^{-3/2}\sum_{\alpha}\int(dp\alpha)[f_{+}(x|p\alpha)a(p\alpha)+f_{-}(x|p\alpha')C^{s}_{\sigma'\sigma}b^{\dagger}(p\alpha)] \tag{8.4}$$

where $\alpha'=(s\sigma'\rho)$, and $(dp\alpha)=d^3p/2E_{\alpha}(\mathbf{p})$ is the Lorentz-invariant volume element. For definiteness we are assuming the particles (operators $a,a^{\dagger}$) to be different from their antiparticles (operators $b,b^{\dagger}$); the case of a strictly neutral Majorana field may be developed along similar lines but appears to be of less importance in physics. The matrix C^s takes into account the slightly different transformation properties of creation and annihilation operators. (Note that we do not require manifest covariance under charge conjugation.) The so-defined field operator $\psi(x)$ satisfies the wave equation

$$(-i\partial\!\!\!/+\mathbf{1})\psi(x)=0 \tag{8.5}$$

and it transforms under the Poincaré group according to

$$U^{-1}(a,g)\psi(x)U(a,g)=D(g)\psi(L^{-1}(x-a)) \tag{8.6}$$

where we have used the property (3.28) of the spinors $w(p\alpha\varepsilon)$.

One of the most fundamental requirements of relativistic quantum field theory, the axiom of local commutativity (microcausality), demands field operators to commute or anticommute at spacelike separations (Bogolubov et al., 1975; Streater and Wightman, 1978). In particular,

$$[\psi(x),\psi^{\dagger}(y)]_{\pm}=0 \qquad \text{for } (x-y)^2<0 \tag{8.7}$$

We can easily evaluate this (anti)commutator of the field $\psi(x)$ with its

Hermitian conjugate $\psi^\dagger(y)$:

$$[\psi_i(x),\psi_k^\dagger(y)]_\pm=(2\pi)^{-3}\sum_\alpha\int(dp\alpha)[u_i(\mathbf{p}\alpha)\otimes u_k^\dagger(\mathbf{p}\alpha)e^{-i_\alpha p(x-y)}$$
$$\pm v_i(\mathbf{p}\alpha)\otimes v_k^\dagger(\mathbf{p}\alpha)e^{i_\alpha p(x-y)}] \tag{8.8}$$

Following arguments similar to those used in the construction of the projection operator $\Gamma^\alpha(q)$, equations (5.29)–(5.34), we find that $u(\mathbf{p}\alpha)\otimes u^\dagger(\mathbf{p}\alpha)$ is a simple matrix polynomial in the 4-vector ${}_\alpha p$,

$$\sum_\sigma u_i(\mathbf{p}\alpha)\otimes u_k^\dagger(\mathbf{p}\alpha)=N_\alpha^2F_+^\alpha({}_\alpha p)_{ik} \tag{8.9a}$$

N_α being the normalization constant (3.22a) of the spinors $u(\mathbf{p}\alpha)$. In the notation of Section 5, this polynomial F_+^α can be defined for any complex 4-vector q^μ off the mass shell:

$$F_+^\alpha(q)_{ik}=D^{A_i}(\{q\}_\alpha)u_i(\alpha)\otimes u_k^\dagger(\alpha)D^{B_k}(\{\tilde{q}\}_\alpha) \tag{8.9b}$$

with the arguments

$$\{q\}_\alpha=q_\mu\sigma^\mu/m_\alpha,\qquad \{\tilde{q}\}_\alpha=\tilde{q}_\mu\sigma^\mu/m_\alpha=q_\mu\tilde{\sigma}^\mu/m_\alpha \tag{8.9c}$$

[Note that in general $\{q\}_\alpha\notin SL(2,C)$; the D matrices, however, are well defined for any 2×2 matrix.] Similarly we compute

$$\sum_\sigma v_i(p\alpha)\otimes v_k^\dagger(p\alpha)=N_\alpha^2F_-^\alpha({}_\alpha p)_{ik} \tag{8.10a}$$

with

$$F_-^\alpha(q)_{ik}=D^{A_i}(\{q\}_\alpha)v_i(\alpha)\otimes v_k^\dagger(\alpha)D^{B_k}(\{\tilde{q}\}_\alpha) \tag{8.10b}$$

These polynomials $F_\pm^\alpha$ are simply related to each other. Using the connection (3.21c) between the positive- and negative-frequency spinors, we find

$$F_-^\alpha(q)_{ik}=(-1)^{2A_k+2B_k}F_+^\alpha(-q)_{ik} \tag{8.11a}$$

that is

$$F_-^\alpha(q)=(-1)^{2s}F_+^\alpha(-q) \tag{8.11b}$$

Hence we obtain for the (anti)commutator

$$[\psi(x),\psi^\dagger(y)]_\pm=-i\sum_\alpha N_\alpha^2F_+^\alpha(i\partial)[\Delta_\alpha^+(x-y)\mp(-1)^{2s}\Delta_\alpha^-(x-y)] \tag{8.12}$$

expressed in terms of the invariant functions $\Delta_k^{\pm}$ of (6.5a). The micro-causality condition can be satisfied if and only if

$$\pm(-1)^{2s}=-1 \tag{8.13}$$

giving the usual connection between spin and statistics. Local commutativity does not require the solutions $f_{\pm}(x|p\alpha)$ to form a complete set. In principle we could have left out certain states α from the spectrum of the wave equation in the definition (8.4) of $\psi(x)$, as microcausality only demands that particles and antiparticles occur with equal strength in the field operators (crossing symmetry).

The field operator $\psi^{\dagger}(x)$ has certain disadvantages: Its transformation law $D^{\dagger}(g)$ will generally not be equivalent to $D^{-1}(g)$, nor will the wave equation satisfied by $\psi^{\dagger}(x)$ be simply related to that of $\psi(x)$. Of more importance for many purposes is the following field operator $\bar{\psi}(x)$ based on the adjoint wave functions $\bar{f}_{\pm}(x|p\alpha)$ described in Section 4:

$$\bar{\psi}(x)=(2\pi)^{-3/2}\sum_{\alpha}\int(dp\alpha)\left[a^{\dagger}(p\alpha)\bar{f}_{+}(x|p\alpha)+b(p\alpha')C^{s}_{\sigma'\sigma}\bar{f}_{-}(x|p\alpha)\right] \tag{8.14}$$

This field operator obeys the adjoint wave equation

$$\bar{\psi}(x)(i\overleftarrow{\partial\!\!\!/}+\mathbf{1})=0 \tag{8.15}$$

and it transforms under the Poincaré group according to

$$U^{-1}(a,g)\bar{\psi}(x)U(a,g)=\bar{\psi}(L^{-1}(x-a))D^{-1}(g) \tag{8.16}$$

We obtain for the (anti)commutator

$$\begin{aligned}\left[\psi(x),\bar{\psi}(x')\right]_{\pm}=(2\pi)^{-3}\sum_{\alpha}\int(dp\alpha)\big[&u(\mathbf{p}\alpha)\otimes\bar{u}(\mathbf{p}\alpha)e^{-i_{\omega}p(x-x')}\\ &\pm(-1)^{2s}v(\mathbf{p}\alpha)\otimes\bar{v}(\mathbf{p}\alpha)e^{i_{\omega}p(x-x')}\big]\end{aligned} \tag{8.17}$$

the factor $(-1)^{2s}$ stemming from the square of the matrix C^s. With the aid of equations (5.43a) and (5.43b) this (anti)commutator can be expressed in terms of the invariant functions introduced in Section 6 [in particular, see equations (6.5), (6.14), and (6.22)],

$$\left[\psi(x),\bar{\psi}(x')\right]_{\pm}=-id(i\partial)\Delta(x-x')=-iS(x-x') \tag{8.18}$$

where again only the usual connection between spin and statistics ensures local commutativity. According to (6.23), the equal-time (anti)commutator is canonical,

$$\left[\psi(x),\bar{\psi}(x')\right]_{\pm}=\beta_0^{-1}\delta(\mathbf{r}-\mathbf{r}') \qquad \text{at } x_0=x_0' \tag{8.19}$$

and hence not more singular at the apex of the light cone than in the case of the Dirac field. This remarkable result holds for any regular wave equation, irrespective of the arbitrarily high spins of the particles involved. For it the completeness relation (4.19) was essential, i.e., that we included in the field operators ψ and $\bar{\psi}$ all the states α from the spectrum of the wave equation.

From the transformation properties (3.8) and (3.14) of the one-particle states $|p\alpha\rangle$ under parity and time reversal we deduce for the creation operators

$$U_P a^\dagger(p\alpha)U_P^{-1}=\eta_\alpha^P a^\dagger(\tilde{p}\alpha)D^s(r) \tag{8.20a}$$

$$A_T a^\dagger(p\alpha)A_T^{-1}=\eta_\alpha^T a^\dagger(\tilde{p}\alpha)D^s(r') \tag{8.20b}$$

and corresponding formulas for the annihilation operators. The antiparticle creation operators $b^\dagger(p\alpha)$ transform analogously; we denote their corresponding phases by $\bar{\eta}_\alpha$. The operation U_C of charge conjugation (unitary in the second quantized theory) is defined as

$$U_C a^\dagger(p\alpha)U_C^{-1}=\eta_\alpha^C b^\dagger(p\alpha) \tag{8.20c}$$

The field operators ψ and $\bar{\psi}$ are found to transform under these discrete symmetry operations as follows:

$$U_P\psi(x)U_P^{-1}=V^{-1}\psi(\tilde{x}) \tag{8.21a}$$

$$U_P\bar{\psi}(x)U_P^{-1}=\bar{\psi}(\tilde{x})V \tag{8.21b}$$

$$A_T\psi(x)A_T^{-1}=W^{-1}\psi(Tx) \tag{8.21c}$$

$$A_T\bar{\psi}(x)A_T^{-1}=\bar{\psi}(Tx)W \tag{8.21d}$$

$$U_C\psi(x)U_C^{-1}=Z\psi^\dagger(x) \tag{8.21e}$$

$$U_C\bar{\psi}(x)U_C^{-1}=(-1)^{2s}\bar{\psi}(x)^{\dagger}Z^{-1} \tag{8.21f}$$

provided the particle and antiparticle phases η_α and $\bar{\eta}_\alpha$ are chosen (up to an overall phase factor)

$$\eta_\alpha^P = \eta_{\alpha+}^P, \qquad \bar{\eta}_\alpha^P = (-1)^{2s}(\eta_\alpha^P)^* \tag{8.22a}$$

$$\eta_\alpha^T = \eta_{\alpha+}^T, \qquad \bar{\eta}_\alpha^T = (\eta_\alpha^T)^* \tag{8.22b}$$

$$\eta_\alpha^C = \eta_{\alpha+}^C, \qquad \bar{\eta}_\alpha^C = (\eta_\alpha^C)^* \tag{8.22c}$$

In these formulas V, W, and Z are the matrices introduced in the preceding Section 7, and in that same section we have also studied the phases $\eta_{\alpha+}$ of the positive-frequency solutions $f_+(x|p\alpha)$. The canonical commutations relations (8.18) will be invariant under these transformations if

$$V^{-1}S(x)V = S(\tilde{x}) \tag{8.23a}$$

$$W^{-1}S(x)W = -S^*(Tx) \tag{8.23b}$$

$$Z^{-1}S(x)Z = S^*(x) \tag{8.23c}$$

These relations are automatically satisfied because of the corresponding properties of the invariant function $\Delta(x)$ and the equations (7.57a)–(7.57c) fulfilled by the Klein–Gordon divisor.

Regular wave equations and the corresponding quantum field theories can be derived from a conventional Lagrangian formalism (Heisenberg and Pauli, 1929, 1930; Wentzel, 1949), starting from the Lagrangian density

$$\mathfrak{L}(x) = \bar{\psi}(i\partial\!\!\!/ - \mathbf{1})\psi(x) \tag{8.24}$$

If we assume $\psi(x)$ to transform under the Poincaré group as $\psi'(x') = D(g)\psi(x)$, and the matrices β^μ to form a 4-vector, $D^{-1}(g)\beta^\mu D(g) = (L\beta)^\mu$, then the field $\bar{\psi}(x)$ has to transform according to $\bar{\psi}'(x') = \bar{\psi}(x)D^{-1}(g)$ in order to obtain a Lorentz-invariant Lagrange density. Varying ψ and $\bar{\psi}$ independently, we arrive at the wave equation (1.5) and the adjoint equation (4.1) as the corresponding Euler–Lagrange equations. The momentum $\pi(x)$ canonically conjugate to $\psi(x)$ is given by

$$\pi(x) = \frac{\partial \mathfrak{L}}{\partial \dot{\psi}} = i\bar{\psi}(x)\beta^0 \tag{8.25}$$

and the canonical equal-time (anti)commutation relation,

$$[\psi(x), \pi(x')]_\pm = i\delta(\mathbf{r} - \mathbf{r}') \qquad \text{at } x_0 = x_0' \tag{8.26}$$

agrees with our formula (8.19) above.

Following standard procedures, we find for the energy–momentum tensor

$$T^{\mu\nu}(x)=i\bar{\psi}(x)\beta^{\mu}\partial^{\nu}\psi(x) \tag{8.27}$$

(or its appropriately symmetrized form), and a corresponding formula for the angular momentum tensor. In particular we obtain for the energy–momentum operator

$$P^{\mu}=\int d^3xT^{0\mu}(x)=i\int d^3x\bar{\psi}(x)\beta^0\partial^{\mu}\psi(x) \tag{8.28}$$

Expanding the fields in terms of creation and annihilation operators, equations (8.4) and (8.14), and using the orthogonality relations (4.23) and (4.24) of the plane wave solutions we get

$$P^{\mu}=\sum_{\alpha}\int(dp\alpha)_{\alpha}p^{\mu}\left[a^{\dagger}(p\alpha)a(p\alpha)+(-1)^{2s}b(p\alpha)b^{\dagger}(p\alpha)\right] \tag{8.29}$$

After normal ordering this operator is positive definite.

Finally, there exists a conserved current

$$j^{\mu}(x)=\bar{\psi}(x)\beta^{\mu}\psi(x), \qquad \partial_{\mu}j^{\mu}=0 \tag{8.30}$$

A simple calculation gives for the total charge $Q=\int d^3xj^0(x)$,

$$Q=\sum_{\alpha}\int(dp\alpha)\left[a^{\dagger}(p\alpha)a(p\alpha)-(-1)^{2s}b(p\alpha)b^{\dagger}(p\alpha)\right] \tag{8.31}$$

After normal ordering we find the usual result that particles and antiparticles have opposite charges.

In closing we want to emphasize that the (anti)commutator function $S(x)$ appearing in (8.18) is equal to the difference of the retarded and advanced Green's functions (6.25a) and (6.25b) of the wave equation, $S=S_R-S_A$. From this it then follows that the ingoing and outgoing fields satisfy the same commutation relations, both within the framework of the Källén–Yang–Feldman equations (Takahashi, 1969; Umezawa, 1956) and the Capri construction (Capri, 1969) for external fields. Hence regular wave equations do not suffer from the inconsistency found by Wightman (1976) for two classes of (irregular) wave equations.

9. DISCUSSION

The results of the last section have been anticlimactic as the quantum field theories based on regular wave equations have turned out to be almost embarrassingly orthodox. Then why was all this not already done thirty years ago? There appear to be various reasons, and most of them have little to do with mathematical virtuosity but more with asking the right questions in the first place. Whenever one attempts to generalize the Dirac equation several choices have to be made. At the risk of some oversimplification the various alternatives are (i) single- or multiple-particle theories, and related to it (ii) subsidiary conditions or completeness of states; (iii) a workable expression for the matrices β^{μ} (implicit algebraic definition or explicit formula); and (iv) an appropriate choice of the scalar product and the adjoint wave function.

From the very beginning we have been deliberately looking for theories that describe a whole spectrum of particles. This constitutes a break with the tradition of introducing for each particle its own wave function, which is then coupled to other fields. Hundreds of investigations have failed to produce a satisfactory theory of single particles with arbitrary spin that is also consistent in the presence of interaction. We should accept the fact that such theories do not exist and that searching for such descriptions may be asking the wrong question. After all, why should we attempt to describe situations that do not occur in nature? "Single particles with any spin" do not exist in the dynamical sense; we only observe multiplets of states with various masses and spins, be it in atomic, nuclear, or particle physics. We have to keep in mind the dual role of relativistic wave functions and wave equations: On the one hand they describe the purely kinematical aspects of free particles under Lorentz transformations, and on the other hand they specify how a particle interacts with other fields. The kinematics of a free particle is uniquely determined by its rest mass and spin, and is most conveniently described in the abstract manner summarized in equations (3.1)–(3.16), independently of the particular choice of the wave function. However, different wave functions will in general be dynamically inequivalent once an interaction is turned on. From the point of view of dynamics there is no unique theory for a given mass and spin, and the correct choice of the wave function can only be found by comparison with experiment. (Even for spin-1/2 we believe it is simply wrong to say that every spin-1/2 particle necessarily has to be described by the Dirac equation.) Experiment indicates an underlying multiplet structure of elementary particles, both in their spectroscopy and dynamics. As our theories ought to have at least something to do with nature, we are almost invariably led to equations that describe whole multiplets of particles.

After having decided on wave equations that contain more than one particle, we must then take seriously the full mass spectra of such equations. Interactions will in general cause transitions between all the states in the spectrum of a wave equation, and we find it only logical to demand that all of these states be physically acceptable. We insist that only real masses occur in the wave equation as tachyons appear at present to be not more than a controversial theoretical speculation. We furthermore demand that there are no "superfluous components" in the wave function, i.e., that all its components are actually used to describe particles of finite mass. This is equivalent to the postulate that all the solutions of the (free)wave equation form a complete set. There we are guided by an analogy with ordinary nonrelativistic quantum mechanics where the eigenfunctions of the Hamiltonian are always required to form a complete set. Dependent components correspond to vanishing eigenvalues of β^0 and formally might be interpreted as belonging to particles of infinite mass. Physically, nothing is gained from such dependent components except complications as seen in the example of Bhabha equations with integer spin (Krajcik and Nieto, 1975). There these unphysical "subsidiary components" have to be eliminated from the physical "particle components" by means of the Sakata–Taketani procedure (Sakata and Taketani, 1940a, b). After this separation of components has been accomplished the discussion runs then parallel to the case of Bhabha equations with half-integral spin. In retrospect it can perhaps be made more plausible why we insist on a complete set of states: The completeness relation (4.19) was essential to obtain the canonical equal-time commutation relation (8.19); otherwise this commutator would be more singular and involve certain derivatives of the Delta function. Besides, wave equations with dependent components are degenerate from the viewpoint of the theory of partial differential equations (Wightman, 1973), and even small perturbations can change the characteristics of such equations. Progress in theoretical physics does not come from a detailed study of pathological cases but from generalizations of those theories that work. To us regular wave equations are the natural generalization of the nonrelativistic Schrödinger equation for atomic spectra. There the completeness of the eigenfunctions of the Hamiltonian is always assumed and theories without this property would be considered pathological indeed.

We have defined a wave equation to be regular if its mass spectrum is physically acceptable and all of its solutions form a complete set. It follows almost by definition that the Dirac equation is the only regular single-particle equation—all other regular equations describe a whole spectrum containing different masses and spins. As in the Schrödinger equation for an atom the various members of the spectrum are intrinsically coupled in their properties. Such wave equations seem to offer a holistic approach to multiplets of particles without the introduction of more elementary substructures. Few of the more general wave equations considered thus far in the literature are regular in the above sense, as most of them have "unnecessary" components corresponding to a singular β_0. It might be interjected that in one of the most successful physical theories, namely, in quantum electrodynamics, the photon is described by the vector potential $A^{\mu}(x)$ which has superfluous components. Although massless particles may necessitate a separate treatment, we believe it is only an approximation to describe the photon by its own field, independently from all the other particles. Ultimately it might be possible to find a wave function that describes the photon together with other particles, in particular the vector mesons.

Here in this article we studied free regular wave equations that describe a finite number of massive particles (however, most of our results can be extended to spectra encompassing an infinite number of states). Except for the transformation law (1.3a) of the matrices β^{μ} under the Lorentz group we saw no need for any further restrictions on the β^{μ}. Standardizing the transformation law $D(g)$ of the wave function as a direct sum of its irreducible components we obtained a workable explicit expression for the matrices β^{μ} in terms of the Clebsch–Gordon coefficients of the rotation group. In practice we found the graphical expression for the β^{μ}, as presented in Figure 1, even more convenient than some ad hoc algebra postulated for these matrices. We did not study the algebra generated by the β^{μ} as we see no point in rediscovering in algebraic disguise the well-known symmetry properties and recursion formulas of Racah and higher recoupling coefficients. Simple algebraic relations will only result for particular transformation laws $D(g)$ and special values of the linkage parameters b_{ik} defined in (2.1). None of the algebras proposed thus far has led to a realistic mass spectrum and we have left the transformation law $D(g)$ and the b_{ik} completely unspecified. They ultimately have to be determined from the nonlinear theory of interaction formulated in terms of the fields ψ and $\bar{\psi}$. Qualitatively the mass spectra of regular wave equations seem to be flexible enough to accomodate the observed spectrum of hadrons and their excited states.

After having standardized the transformation law $D(g)$ of the wave function we could in general not impose any reality requirements on the matrices β^{μ}, like a Hermitian β^{0}. (Assuming a real mass spectrum and a complete set of states implies of course β_0 to be equivalent to $\beta_0^{\dagger}$.) In particular we did not require the existence of a hermitizing matrix η such that $\beta_{\mu}^{\dagger}=\eta\beta_{\mu}\eta^{-1}$. If there is any significant difference between general regular wave equations and the special case of the Dirac equation, it lies in the scalar product and the adjoint wave functions. With few exceptions (Weaver et al., 1964; Hurley, 1974; Velo and Wightman, 1978) the scalar product has been defined with the help of a hermitizing matrix or by assuming that there is a simple local connection—one that does not involve derivatives—between $\bar{\psi}(x)$ and the complex conjugate wave function. It appears one has to be less axiomatic about the way the scalar product is defined [in this connection also see the discussion of Wightman

(Velo and Wightman, 1978[3])]. The standard approach has led to the well-known difficulties of negative energies (Harris, 1955; Chang, 1966; Tung, 1967) or an indefinite metric (Krajcik and Nieto, 1976); this invariably happens whenever there occur states with different intrinsic parities in the spectrum of the wave equation. However, these difficulties are solely caused by an inappropriate definition of the scalar product. We do not agree with the conclusion reached by Krajcik and Nieto (1977) that an indefinite metric is essential for consistent theories of higher spin. There is no reason at all why the scalar product and the adjoint wave function have to be constructed in that particular and restricted way, and we have shown that there actually exists a scalar product that is free of all these difficulties: It fulfills the usual requirements (4.26)–(4.29); within the positive definite metric (8.3) it leads to positive energies, equation (8.29), and the canonical commutation relations (8.18) and (8.19) without destroying the relation $S(x)=S_R(x)-S_A(x)$.

Regular wave equations seem to offer a basis for a consistent description of (multiplets of) particles with any spin: They lead to quantum field theories with positive definite metric and energy, canonical commutation relations, and propagators without contact terms. Although here we have only considered noninteracting fields, it is apparent by now (and will be shown in detail somewhere else) that regular wave equations are free of at least the worst inconsistencies that have plagued higher spin theories thus far. As we do not impose any constraints we are in no danger of losing any in the presence of interaction. Regular wave equations will be shown to be causal, at least for minimal coupling. As indicated at the end of the last section, the ingoing and outgoing field operators satisfy the same commutation relations as required for a consistent particle interpretation. We expect that the existence of a unitary S matrix for regular wave equations in external fields can be proved analogously to the case of the Dirac equation (Capri, 1969). Considering the explicit form of the propagators one might even speculate that the corresponding quantum field theories are renormalizable.

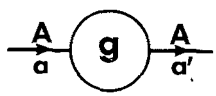
APPENDIX A: GRAPHICAL CALCULUS FOR D MATRICES AND CLEBSCH–GORDAN COEFFICIENTS

We have assumed the transformation law $D(g)$ of the wave function to be given in completely reduced standard form, i.e., as the direct sum of its irreducible components, $D(g)=\Sigma\oplus D^i(g)$. These matrices $D^i(g)=D^{A_i}(g)\otimes D^{B_i}(g)^{\dagger-1}$, together with the appropriate Clebsch–Gordan coefficients, constitute the fundamental entities from which all physical quantities are built, like the matrices β^μ, the plane wave solutions $f_\pm(x|p\alpha)$, the

[3]See in particular pp. 36–45.

Klein–Gordon divisor, and the ensuing invariant functions. Therefore, in many calculations we shall encounter numerous Clebsch–Gordan coefficients and D matrices of various arguments, with a summation over all the internal magnetic quantum numbers. For that purpose we want to sketch a graphical representation of D matrices and Clebsch–Gordan coefficients, first developed in connection with the atomic and nuclear shell models (Yutsis et al., 1962; El-Baz and Castel, 1972), which makes actual calculations with the general wave equation (1.5) feasible.

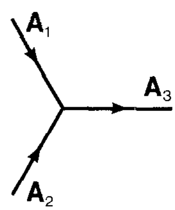
We represent the matrix elements $D^A(g)_{aa'}$ of the irreducible representation D^A of $SL(2,C)$ by the symbol

$D^A(g)_{aa'} =$ 

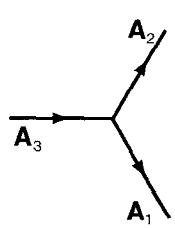
(A.1)

To distinguish a matrix from its transpose, our formalism is based on directed lines: Ingoing (outgoing) lines refer to row (column) indices of matrices, and we shall usually suppress the magnetic quantum numbers a and a'. The circle will be exclusively used for the representation matrices D^A; all other matrices will be depicted by different symbols (as, for example, the Pauli matrices σ^μ in Figure 1).

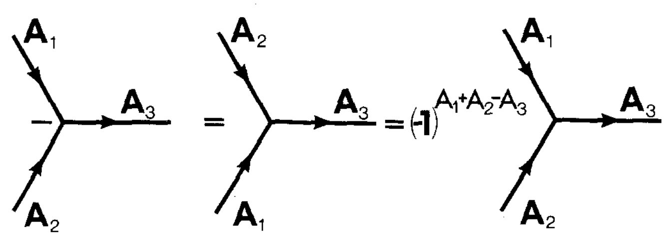
We picture the (normalized) Clebsch–Gordan coefficients by vertices with two ingoing (outgoing) and one outgoing (ingoing) line:

$[A_3]^{-1/2}\langle A_1a_1A_2a_2|A_3a_3\rangle =$ 

(A.2)

$= [A_3]^{-1/2}\langle A_3a_3|A_1a_1A_2a_2\rangle =$ 

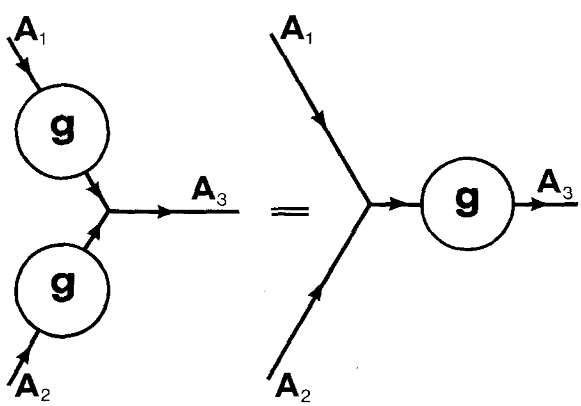
The normalization factor $[A]^{-1/2}$, with $[A]=2A+1$, has been chosen to get simple symmetry properties of these graphs (Biritz, 1975a; Dürr and Wagner, 1968). The order in which the angular momenta A_1 and A_2 are coupled together is usually determined by the rule that going around the vertex from line A_1 to line A_2 shall correspond to a rotation in the positive sense. However, for topological reasons, it is sometimes convenient to follow just the opposite convention of a negative rotation; this we shall indicate by a minus sign written next to that vertex. Because of the symmetry properties of the Clebsch–Gordan coefficients, these two conventions merely differ by a phase factor:



(A.3)

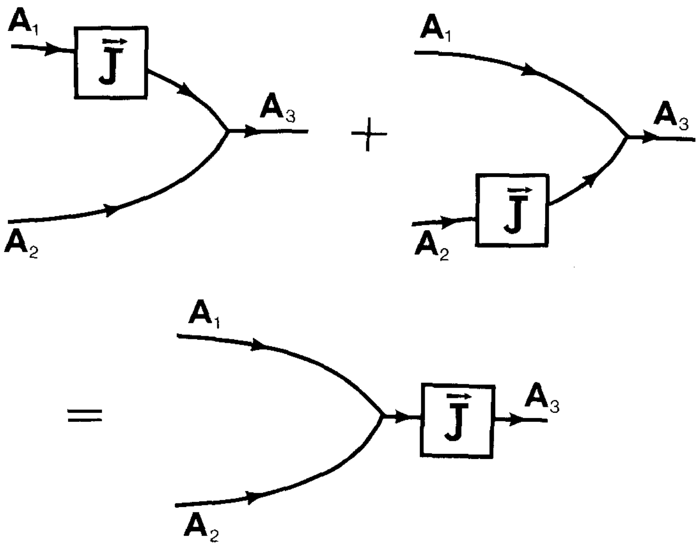
There are further simple rules (Biritz, 1975a) for changing the orientation of external and internal lines, which we do not need to elaborate here.

The fundamental relation between Clebsch–Gordan coefficients and the irreducible representations $D^A(g)$ is graphically expressed as



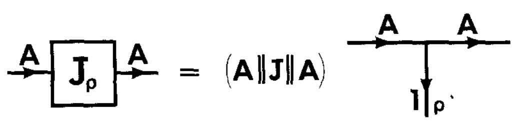
(A.4)

Differentiating this equation, we obtain the following identity for the angular momentum matrices **J**:



(A.5)

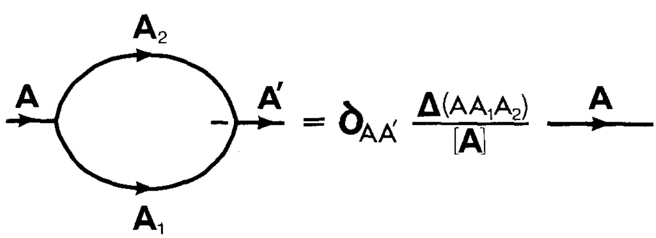
As a special example of the Wigner–Eckart theorem, the elements of the angular momentum matrices $(\mathbf{J}^A)_{aa'}$ can be expressed in terms of Clebsch–Gordan coefficients. Graphically, we find for the spherical vector components $J_\rho^A, \rho = 1, 0, -1$,



(A.6)

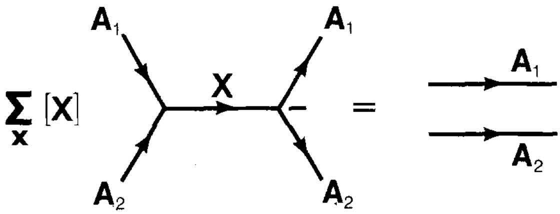
with the usual abbreviation $(A\|J\|A) = [A(A+1)(2A+1)]^{1/2}$ for the reduced angular momentum matrix element.

The orthogonality and completeness relations of the Clebsch–Gordan coefficients are depicted as



(A.7)

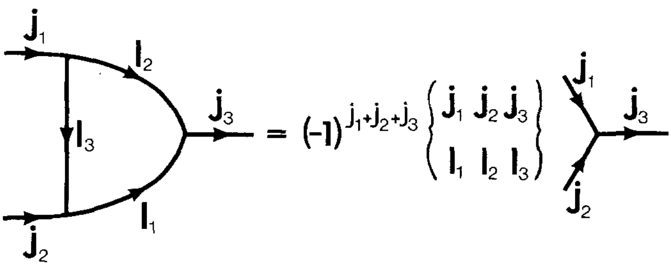
and



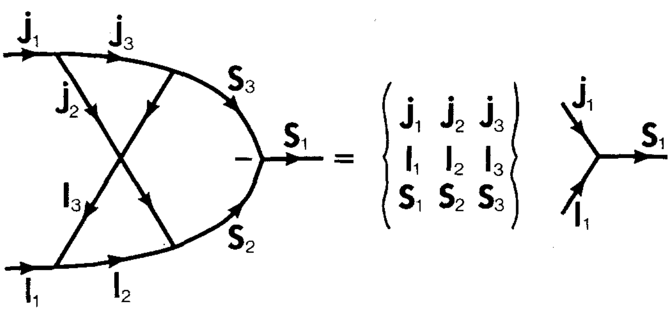
(A.8)

The single lines on the right-hand side of these equations represent the appropriate unit matrices; $\Delta(A_1A_2A_3)=1$ if the three angular momenta satisfy the triangle inequality, $\Delta=0$ otherwise. For internal lines a sum over the corresponding magnetic quantum numbers is always to be understood.

Finally, we give a graphical definition of the $6j$ (Racah) and $9j$ symbols:



(A.9)



(A.10)

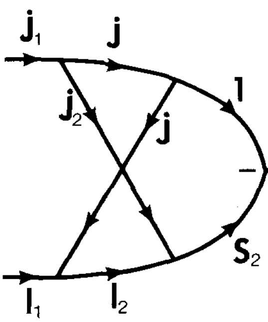
It turns out that the orientation of the internal l_3 line in (A.9) is irrelevant, as are the orientations of the j_2 and l_3 lines in (A.10).

Such simple graphical rules allow an almost effortless handling of higher recoupling coefficients, including their correct phase factors; an application of these rules will be given in Appendix B.

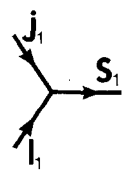
APPENDIX B: A RECURSION FORMULA FOR CERTAIN 9*j* SYMBOLS

In Section 2 we derived from the commutation relation (2.11b) of the β matrices the connection (2.15) between the reduced matrix elements of β and β_0, that is between 9*j* and 6*j* symbols. This relation is a particular example of a more general recursion formula which holds for certain 9*j* symbols; it is based on the simple property (A.5) of the angular momentum matrices and their graphical expression (A.6).

Consider the 9*j* symbols defined in (A.10) for the special case of $j_3 = l_3 = j$, and $s_3 = 1$:

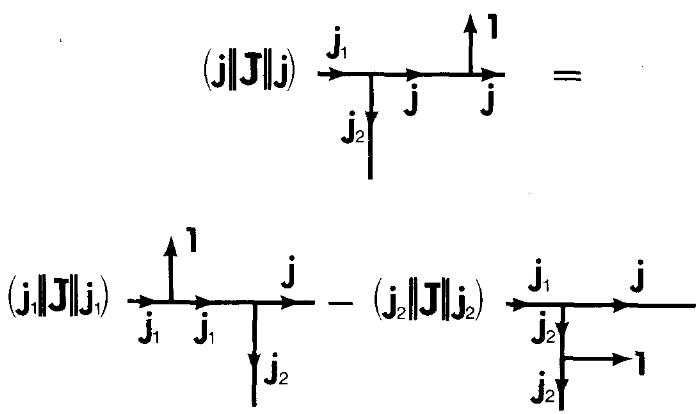


$$= \begin{Bmatrix} j_1 & j_2 & j \\ l_1 & l_2 & j \\ s_1 & s_2 & 1 \end{Bmatrix}$$



(B.1)

In the graph on the left-hand side there occur the two vertices coupling the $(j_1 j_2 j)$ and $(jj1)$ lines. According to (A.5) and (A.6) we have the identity



(B.2)

This identity (B.2) allows us to recouple the $s_3 = 1$ line from the j line onto the j_1 and j_2 lines. Therefore the graph in (B.1) can be expressed as a sum of two graphs corresponding to the recoupled $s_3 = 1$ line. Each of these two new graphs is found to contain two internal triangles, which, according to the rule (A.9), can be collapsed to a point times a Racah coefficient. Collecting all normalization factors and phases we obtain the recursion formula

$$(j\|J\|j)\begin{Bmatrix} j_1 & j_2 & j \\ l_1 & l_2 & j \\ s_1 & s_2 & 1 \end{Bmatrix}$$

$$=(-1)^{l_1+j-l_2}(-1)^{s_1-s_2}(j_2\|J\|j_2)\begin{Bmatrix} j_1 & j_2 & j \\ l_2 & l_1 & s_1 \end{Bmatrix}\begin{Bmatrix} j_2 & l_2 & s_2 \\ s_1 & 1 & j_2 \end{Bmatrix}$$

$$-(-1)^{j_2+j-j_1}(j_1\|J\|j_1)\begin{Bmatrix} j_1 & j_2 & j \\ l_2 & l_1 & s_2 \end{Bmatrix}\begin{Bmatrix} j_1 & l_1 & s_1 \\ s_2 & 1 & j_1 \end{Bmatrix} \qquad \text{(B.3)}$$

where again $(j\|J\|j)=[j(j+1)(2j+1)]^{1/2}$. The relation (2.15) corresponds to the special case where $j=1/2$.

We take this opportunity to correct some unfortunate mistakes in the literature. Setting $s_1=s_2=s$ in (B.3), we find the formula

$$\begin{Bmatrix} j_1 & j_2 & j \\ l_1 & l_2 & j \\ s & s & 1 \end{Bmatrix}$$

$$=(-1)^{l_1+s+j_2+j}\frac{j_1(j_1+1)-j_2(j_2+1)-l_1(l_1+1)+l_2(l_2+1)}{2[s(s+1)(2s+1)j(j+1)(2j+1)]^{1/2}}\begin{Bmatrix} j_1 & j_2 & j \\ l_2 & l_1 & s \end{Bmatrix} \qquad \text{(B.4)}$$

In some books (deShalit and Talmi, 1963; deShalit and Feshbach, 1974) an expression is given for this $9j$ symbol where both the normalization and the phase factor are wrong. [The formulas given in deShalit and Talmi (1963) and deShalit and Feshbach (1974) are easily seen to be incorrect by setting $j_1=s$, $l_1=0$, $j_2=s-1/2$, $l_2=1/2$, $j=1/2$, and computing the ensuing $9j$ symbol directly.] As far as we can tell, this error can be traced to the wrong formula (3.24) of Rotenberg, Metropolis, Bivens, and Wooten (1959), which in turn is based on the wrong equation (3.23) of the same reference: In the third Racah coefficient on the right-hand side of equation (3.23) of Rotenberg et al. (1959), the triplet $(ij\mu)$ has to be replaced by $(hj\mu)$. [The correct formula can be found in the original paper of Arima, Horie, and Tanabe (1954), which is reprinted in Biedenharn and van Dam (1965).]

We have checked our formula (B.3) with explicit expressions of $9j$ symbols as given, for example, in Sobel'man (1972)—and vice versa: There is an overall minus sign missing in the second entry, that for $l=j+1/2, l'=j'-1/2$, of Table 60 on p.181 of Ref. Sobel'man (1972).

APPENDIX C: TWO SPECIAL RACAH COEFFICIENTS

For completeness we list here the two types of Racah coefficients that may occur in the reduced mass matrix $\Lambda(s)$ defined in Section 2, and from which the mass spectrum of the wave equation is determined. We find from standard tables (deShalit and Talmi, 1963; Sobel'man, 1972)

$$\begin{Bmatrix} A & B & s \\ B+1/2 & A+1/2 & 1/2 \end{Bmatrix} = (-1)^{A+B+s+1}\left[\frac{(A+B+1)(A+B+2)-s(s+1)}{(2A+1)(2A+2)(2B+1)(2B+2)}\right]^{1/2} \tag{C.1}$$

$$\begin{Bmatrix} A & B & s \\ B+1/2 & A-1/2 & 1/2 \end{Bmatrix} = (-1)^{A+B+s}\left[\frac{s(s+1)-(A-B-1)(A-B)}{2A(2A+1)(2B+1)(2B+2)}\right]^{1/2} \tag{C.2}$$

where the various angular momenta are assumed to satisfy the triangle inequalities appropriate for a Racah coefficient. We have slightly rearranged the numerator to show more clearly the spin dependence of these coefficients: (C.1) is a decreasing function of spin whereas (C.2) is increasing. In general both of these two types of Racah coefficients will contribute to the mass spectrum of the wave equation.

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